CHAPTER 8
Sequences, Series, and Probability

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Section 8.1 Sequences and Series

- Given the general nth term in a sequence, you should be able to find, or list, some of the terms.
- You should be able to find an expression for the nth term of a sequence.
- You should be able to use and evaluate factorials.
- You should be able to use sigma notation for a sum.

Vocabulary Check

1. infinite sequence
2. terms
3. finite
4. recursively
5. factorial
6. summation notation
7. index, upper limit, lower limit
8. series
9. nth partial sum

1. $a_n = 2n + 5$

$a_1 = 2(1) + 5 = 7$

$a_2 = 2(2) + 5 = 9$

$a_3 = 2(3) + 5 = 11$

$a_4 = 2(4) + 5 = 13$

$a_5 = 2(5) + 5 = 15$

2. $a_n = 4n - 7$

$a_1 = 4(1) - 7 = -3$

$a_2 = 4(2) - 7 = 1$

$a_3 = 4(3) - 7 = 5$

$a_4 = 4(4) - 7 = 9$

$a_5 = 4(5) - 7 = 13$

3. $a_n = 2^n$

$a_1 = 2^1 = 2$

$a_2 = 2^2 = 4$

$a_3 = 2^3 = 8$

$a_4 = 2^4 = 16$

$a_5 = 2^5 = 32$

4. $a_n = \left(\frac{1}{2}\right)^n$

$a_1 = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$

$a_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$a_3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$a_4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

$a_5 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

5. $a_n = \left(-\frac{1}{2}\right)^n$

$a_1 = \left(-\frac{1}{2}\right)^1 = -\frac{1}{2}$

$a_2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

$a_3 = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$

$a_4 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16}$

$a_5 = \left(-\frac{1}{2}\right)^5 = -\frac{1}{32}$

6. $a_n = (-2)^n$

$a_1 = (-2)^1 = -2$

$a_2 = (-2)^2 = 4$

$a_3 = (-2)^3 = -8$

$a_4 = (-2)^4 = 16$

$a_5 = (-2)^5 = -32$

7. $a_n = \dfrac{n+1}{n}$

$a_1 = \dfrac{1+1}{1} = 2$

$a_2 = \dfrac{3}{2}$

$a_3 = \dfrac{4}{3}$

$a_4 = \dfrac{5}{4}$

$a_5 = \dfrac{6}{5}$

8. $a_n = \dfrac{n}{n+1}$

$a_1 = \dfrac{1}{1+1} = \dfrac{1}{2}$

$a_2 = \dfrac{2}{2+1} = \dfrac{2}{3}$

$a_3 = \dfrac{3}{3+1} = \dfrac{3}{4}$

$a_4 = \dfrac{4}{4+1} = \dfrac{4}{5}$

$a_5 = \dfrac{5}{5+1} = \dfrac{5}{6}$

9. $a_n = \dfrac{n}{n^2+1}$

$a_1 = \dfrac{1}{1^2+1} = \dfrac{1}{2}$

$a_2 = \dfrac{2}{2^2+1} = \dfrac{2}{5}$

$a_3 = \dfrac{3}{3^2+1} = \dfrac{3}{10}$

$a_4 = \dfrac{4}{4^2+1} = \dfrac{4}{17}$

$a_5 = \dfrac{5}{5^2+1} = \dfrac{5}{26}$

10. $a_n = \dfrac{2n}{n+1}$

$a_1 = \dfrac{2(1)}{1+1} = 1$

$a_2 = \dfrac{4}{3}$

$a_3 = \dfrac{6}{4} = \dfrac{3}{2}$

$a_4 = \dfrac{8}{5}$

$a_5 = \dfrac{10}{6} = \dfrac{5}{3}$

11. $a_n = \dfrac{1+(-1)^n}{n}$

$a_1 = 0$

$a_2 = \dfrac{2}{2} = 1$

$a_3 = 0$

$a_4 = \dfrac{2}{4} = \dfrac{1}{2}$

$a_5 = 0$

12. $a_n = \dfrac{1+(-1)^n}{2n}$

$a_1 = \dfrac{1-1}{2} = 0$

$a_2 = \dfrac{1+1}{2(2)} = \dfrac{1}{2}$

$a_3 = \dfrac{1-1}{2(3)} = 0$

$a_4 = \dfrac{1+1}{2(4)} = \dfrac{1}{4}$

$a_5 = \dfrac{1-1}{2(5)} = 0$

13. $a_n = 1 - \dfrac{1}{2^n}$

$a_1 = 1 - \dfrac{1}{2^1} = \dfrac{1}{2}$

$a_2 = 1 - \dfrac{1}{2^2} = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

$a_3 = 1 - \dfrac{1}{2^3} = \dfrac{7}{8}$

$a_4 = 1 - \dfrac{1}{2^4} = \dfrac{15}{16}$

$a_5 = 1 - \dfrac{1}{2^5} = \dfrac{31}{32}$

14. $a_n = \dfrac{3^n}{4^n}$

$a_1 = \dfrac{3^1}{4^1} = \dfrac{3}{4}$

$a_2 = \dfrac{3^2}{4^2} = \dfrac{9}{16}$

$a_3 = \dfrac{3^3}{4^3} = \dfrac{27}{64}$

$a_4 = \dfrac{3^4}{4^4} = \dfrac{81}{256}$

$a_5 = \dfrac{3^5}{4^5} = \dfrac{243}{1024}$

15. $a_n = \dfrac{1}{n^{3/2}}$

$a_1 = \dfrac{1}{1} = 1$

$a_2 = \dfrac{1}{2^{3/2}}$

$a_3 = \dfrac{1}{3^{3/2}}$

$a_4 = \dfrac{1}{4^{3/2}} = \dfrac{1}{8}$

$a_5 = \dfrac{1}{5^{3/2}}$

16. $a_n = \dfrac{1}{\sqrt{n}}$

$a_1 = 1$

$a_2 = \dfrac{1}{\sqrt{2}}$

$a_3 = \dfrac{1}{\sqrt{3}}$

$a_4 = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$

$a_5 = \dfrac{1}{\sqrt{5}}$

17. $a_n = \dfrac{(-1)^n}{n^2}$

$a_1 = \dfrac{-1}{1} = -1$

$a_2 = \dfrac{1}{4}$

$a_3 = \dfrac{-1}{9}$

$a_4 = \dfrac{1}{16}$

$a_5 = \dfrac{-1}{25}$

18. $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$

$a_1 = (-1)^1\dfrac{1}{1+1} = -\dfrac{1}{2}$

$a_2 = (-1)^2\dfrac{2}{1+2} = \dfrac{2}{3}$

$a_3 = (-1)^3\dfrac{3}{3+1} = -\dfrac{3}{4}$

$a_4 = (-1)^4\dfrac{4}{4+1} = \dfrac{4}{5}$

$a_5 = (-1)^5\dfrac{5}{5+1} = -\dfrac{5}{6}$

19. $a_n = (2n-1)(2n+1)$

$a_1 = (1)(3) = 3$

$a_2 = (3)(5) = 15$

$a_3 = (5)(7) = 35$

$a_4 = (7)(9) = 63$

$a_5 = (9)(11) = 99$

20. $a_n = n(n-1)(n-2)$

$a_1 = 1(1-1)(1-2) = 0$

$a_2 = 2(2-1)(2-2) = 0$

$a_3 = 3(3-1)(3-2) = 6$

$a_4 = 4(4-1)(4-2) = 24$

$a_5 = 5(5-1)(5-2) = 60$

21. $a_{25} = (-1)^{25}[3(25) - 2] = -73$

22. $a_{16} = (-1)^{15}[16(15)] = -240$

23. $a_{10} = \dfrac{10^2}{10^2+1} = \dfrac{100}{101}$

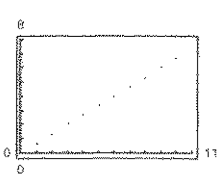
24. $a_n = \dfrac{n^2}{2n+1}$

$a_5 = \dfrac{5^2}{2(5)+1} = \dfrac{25}{11}$

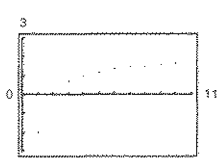
25. $a_6 = \dfrac{2^6}{2^6+1} = \dfrac{64}{65}$

26. $a_7 = \dfrac{2^{7+1}}{2^7+1} = \dfrac{2^8}{2^7+1} = \dfrac{256}{129}$

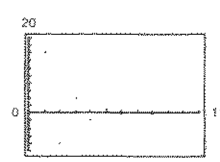
27. $a_n = \dfrac{2}{3}n$



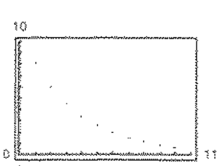
28. $a_n = 2 - \dfrac{4}{n}$



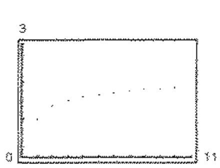
29. $a_n = 16(-0.5)^{n-1}$



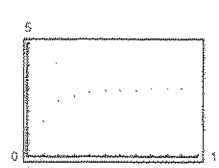
30. $a_n = 8(0.75)^{n-1}$



31. $a_n = \dfrac{2n}{n+1}$



32. $a_n = \dfrac{3n^2}{n^2+1}$



33. $a_n = 2(3n - 1) + 5$

n	1	2	3	4	5	6	7	8	9	10
a_n	9	15	21	27	33	39	45	51	57	63

34. $a_n = 2n(n + 1)(n + 2)$

n	1	2	3	4	5	6	7	8	9	10
a_n	12	48	120	240	420	672	1008	1440	1980	2640

35. $a_n = 1 + \frac{n + 1}{n}$

n	1	2	3	4	5	6	7	8	9	10
a_n	3	2.5	2.33	2.25	2.2	2.17	2.14	2.13	2.11	2.1

36. $a_n = \frac{4n^2}{(n + 2)}$

n	1	2	3	4	5	6	7	8	9	10
a_n	$\frac{4}{3}$	4	7.2	10.67	14.29	18	21.78	25.6	29.45	33.33

37. $a_n = (-1)^n + 1$

n	1	2	3	4	5	6	7	8	9	10
a_n	0	2	0	2	0	2	0	2	0	2

38. $a_n = (-1)^{n+1} + 1$

n	1	2	3	4	5	6	7	8	9	10
a_n	2	0	2	0	2	0	2	0	2	0

39. $a_n = \frac{8}{n + 1}$

$a_n \to 0$ as $n \to \infty$

$a_1 = 4,\ a_{10} = \frac{8}{11}$

Matches graph (c).

40. $a_n = \frac{8n}{n + 1}$

$a_n \to 8$ as $n \to \infty$

$a_1 = 4,\ \ a_4 = \frac{8(4)}{5} = \frac{32}{5}$

Matches graph (b).

41. $a_n = 4(0.5)^{n-1}$

$a_n \to 0$ as $n \to \infty$

$a_1 = 4,\ a_{10} \approx 0.008$

Matches graph (d).

42. $a_n = \dfrac{4^n}{n!}$

$a_n \to 0$ as $n \to \infty$

$a_1 = 4,\ a_4 = \dfrac{4^4}{4!} = \dfrac{256}{24} = 10\dfrac{2}{3}$

Matches graph (a).

43. $1, 4, 7, 10, 13, \ldots$

$a_n = 1 + (n-1)3 = 3n - 2$

44. $3, 7, 11, 15, 19, \ldots$

$a_n = 4n - 1$

45. $0, 3, 8, 15, 24, \ldots$

$a_n = n^2 - 1$

46. $1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \dfrac{1}{25}, \ldots$

$a_n = \dfrac{1}{n^2}$

47. $\dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}, \dfrac{6}{7}, \ldots$

$a_n = \dfrac{n+1}{n+2}$

48. $\dfrac{2}{1}, \dfrac{3}{3}, \dfrac{4}{5}, \dfrac{5}{7}, \dfrac{6}{9}, \ldots$

$a_n = \dfrac{n+1}{2n-1}$

49. $\dfrac{1}{2}, \dfrac{-1}{4}, \dfrac{1}{8}, \dfrac{-1}{16}, \ldots$

$a_n = \dfrac{(-1)^{n+1}}{2^n}$

50. $\dfrac{1}{3}, \dfrac{-2}{9}, \dfrac{4}{27}, \dfrac{-8}{81}, \ldots$

$a_n = \dfrac{(-1)^{n+1}2^{n-1}}{3^n} = \dfrac{(-2)^{n-1}}{3^n}$

51. $1 + \dfrac{1}{1}, 1 + \dfrac{1}{2}, 1 + \dfrac{1}{3}, 1 + \dfrac{1}{4}, 1 + \dfrac{1}{5}, \ldots$

$a_n = 1 + \dfrac{1}{n}$

52. $1 + \dfrac{1}{2}, 1 + \dfrac{3}{4}, 1 + \dfrac{7}{8}, 1 + \dfrac{15}{16}, 1 + \dfrac{31}{32}, \ldots$

$a_n = 1 + \dfrac{2^n - 1}{2^n}$

53. $1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, \ldots$

$a_n = \dfrac{1}{n!}$

54. $1, 2, \dfrac{2^2}{2}, \dfrac{2^3}{6}, \dfrac{2^4}{24}, \dfrac{2^5}{120}, \ldots$

$a_n = \dfrac{2^{n-1}}{(n-1)!}$

55. $1, 3, 1, 3, 1, 3, \ldots$

$a_n = 2 + (-1)^n$

56. $1, -1, 1, -1, 1, -1, \ldots$

$a_n = (-1)^{n+1}$

57. $a_1 = 28$ and $a_{k+1} = a_k - 4$

$a_1 = 28$

$a_2 = a_1 - 4 = 28 - 4 = 24$

$a_3 = a_2 - 4 = 24 - 4 = 20$

$a_4 = a_3 - 4 = 20 - 4 = 16$

$a_5 = a_4 - 4 = 16 - 4 = 12$

58. $a_1 = 15,\quad a_{k+1} = a_k + 3$

$a_1 = 15$

$a_2 = a_1 + 3 = 15 + 3 = 18$

$a_3 = a_2 + 3 = 18 + 3 = 21$

$a_4 = a_3 + 3 = 21 + 3 = 24$

$a_5 = a_4 + 3 = 24 + 3 = 27$

59. $a_1 = 3$ and $a_{k+1} = 2(a_k - 1)$

$a_1 = 3$

$a_2 = 2(a_1 - 1) = 2(3 - 1) = 4$

$a_3 = 2(a_2 - 1) = 2(4 - 1) = 6$

$a_4 = 2(a_3 - 1) = 2(6 - 1) = 10$

$a_5 = 2(a_4 - 1) = 2(10 - 1) = 18$

60. $a_1 = 32,\quad a_{k+1} = \frac{1}{2}a_k$

$a_1 = 32$

$a_2 = \frac{1}{2}a_1 = \frac{1}{2}(32) = 16$

$a_3 = \frac{1}{2}a_2 = \frac{1}{2}(16) = 8$

$a_4 = \frac{1}{2}a_3 = \frac{1}{2}(8) = 4$

$a_5 = \frac{1}{2}a_4 = \frac{1}{2}(4) = 2$

61. $a_1 = 6$ and $a_{k+1} = a_k + 2$

$a_1 = 6$

$a_2 = a_1 + 2 = 6 + 2 = 8$

$a_3 = a_2 + 2 = 8 + 2 = 10$

$a_4 = a_3 + 2 = 10 + 2 = 12$

$a_5 = a_4 + 2 = 12 + 2 = 14$

In general, $a_n = 2n + 4$.

62. $a_1 = 25,\ \ a_{k+1} = a_k - 5$

$a_1 = 25$

$a_2 = a_1 - 5 = 25 - 5 = 20$

$a_3 = a_2 - 5 = 20 - 5 = 15$

$a_4 = a_3 - 5 = 15 - 5 = 10$

$a_5 = a_4 - 5 = 10 - 5 = 5$

In general, $a_n = 30 - 5n$.

63. $a_1 = 81$ and $a_{k+1} = \frac{1}{3}a_k$

$a_1 = 81$

$a_2 = \frac{1}{3}a_1 = \frac{1}{3}(81) = 27$

$a_3 = \frac{1}{3}a_2 = \frac{1}{3}(27) = 9$

$a_4 = \frac{1}{3}a_3 = \frac{1}{3}(9) = 3$

$a_5 = \frac{1}{3}a_4 = \frac{1}{3}(3) = 1$

In general, $a_n = 81\left(\frac{1}{3}\right)^{n-1} = 81(3)\left(\frac{1}{3}\right)^n = \frac{243}{3^n}$.

64. $a_1 = 14,\ \ a_{k+1} = (-2)a_k$

$a_1 = 14$

$a_2 = (-2)a_1 = (-2)(14) = -28$

$a_3 = (-2)a_2 = (-2)(-28) = 56$

$a_4 = (-2)a_3 = (-2)(56) = -112$

$a_5 = (-2)(a_4) = (-2)(-112) = 224$

In general, $a_n = 14(-2)^{n-1}$.

65. $a_n = \frac{1}{n!}$

$a_0 = \frac{1}{0!} = 1$

$a_1 = \frac{1}{1!} = 1$

$a_2 = \frac{1}{2}$

$a_3 = \frac{1}{3!} = \frac{1}{6}$

$a_4 = \frac{1}{4!} = \frac{1}{24}$

66. $a_n = \frac{1}{(n+1)!}$

$a_0 = \frac{1}{1!} = 1$

$a_1 = \frac{1}{2!} = \frac{1}{2}$

$a_2 = \frac{1}{3!} = \frac{1}{6}$

$a_3 = \frac{1}{4!} = \frac{1}{24}$

$a_4 = \frac{1}{5!} = \frac{1}{120}$

67. $a_n = \frac{n!}{2n+1}$

$a_0 = \frac{0!}{1} = 1$

$a_1 = \frac{1!}{2+1} = \frac{1}{3}$

$a_2 = \frac{2!}{4+1} = \frac{2}{5}$

$a_3 = \frac{3!}{6+1} = \frac{6}{7}$

$a_4 = \frac{4!}{8+1} = \frac{24}{9} = \frac{8}{3}$

68. $a_n = \dfrac{n^2}{(n+1)!}$

$a_0 = 0$

$a_1 = \dfrac{1}{2}$

$a_2 = \dfrac{2^2}{3!} = \dfrac{2}{3}$

$a_3 = \dfrac{3^2}{4!} = \dfrac{9}{24} = \dfrac{3}{8}$

$a_4 = \dfrac{16}{5!} = \dfrac{16}{120} = \dfrac{2}{15}$

69. $a_n = \dfrac{(-1)^{2n}}{(2n)!}$

$a_0 = \dfrac{(-1)^0}{0!} = 1$

$a_1 = \dfrac{(-1)^2}{2!} = \dfrac{1}{2}$

$a_2 = \dfrac{(-1)^4}{4!} = \dfrac{1}{24}$

$a_3 = \dfrac{(-1)^6}{6!} = \dfrac{1}{720}$

$a_4 = \dfrac{(-1)^8}{8!} = \dfrac{1}{40{,}320}$

70. $a_n = \dfrac{(-1)^{2n+1}}{(2n+1)!}$

$a_0 = \dfrac{-1^1}{1!} = -1$

$a_1 = \dfrac{(-1)^3}{3!} = \dfrac{-1}{6}$

$a_2 = \dfrac{-1}{5!} = \dfrac{-1}{120}$

$a_3 = \dfrac{-1}{7!} = \dfrac{-1}{5040}$

$a_4 = \dfrac{-1}{9!} = \dfrac{-1}{362{,}880}$

71. $\dfrac{2!}{4!} = \dfrac{2!}{4 \cdot 3 \cdot 2!} = \dfrac{1}{12}$

72. $\dfrac{5!}{7!} = \dfrac{5!}{7(6)(5!)} = \dfrac{1}{42}$

73. $\dfrac{12!}{4!8!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8!}{4!8!}$

$= \dfrac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2} = 495$

74. $\dfrac{10!\,3!}{4!\,6!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6! \cdot 3!}{4 \cdot 3! \cdot 6!}$

$= \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{4} = 1260$

75. $\dfrac{(n+1)!}{n!} = \dfrac{(n+1)n!}{n!} = n+1$

76. $\dfrac{(n+2)!}{n!} = \dfrac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$

77. $\dfrac{(2n-1)!}{(2n+1)!} = \dfrac{(2n-1)!}{(2n+1)(2n)(2n-1)!}$

$= \dfrac{1}{2n(2n+1)}$

78. $\dfrac{(2n+2)!}{(2n)!} = \dfrac{(2n+2)(2n+1)(2n)!}{(2n)!}$

$= (2n+2)(2n+1)$

79. $\displaystyle\sum_{i=1}^{5}(2i+1) = (2+1) + (4+1) + (6+1) + (8+1) + (10+1) = 35$

80. $\displaystyle\sum_{i=1}^{6}(3i-1) = (3 \cdot 1 - 1) + (3 \cdot 2 - 1) + (3 \cdot 3 - 1) + (3 \cdot 4 - 1) + (3 \cdot 5 - 1) + (3 \cdot 6 - 1) = 57$

81. $\displaystyle\sum_{k=1}^{4} 10 = 10 + 10 + 10 + 10 = 40$

82. $\displaystyle\sum_{k=1}^{5} 6 = 6 + 6 + 6 + 6 + 6 = 30$

83. $\displaystyle\sum_{i=0}^{4} i^2 = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 30$

84. $\displaystyle\sum_{k=0}^{5} 3i^2 = 3\sum_{i=0}^{5} i^2$

$= 3(0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2) = 165$

85. $\displaystyle\sum_{k=0}^{3} \frac{1}{k^2+1} = \frac{1}{1} + \frac{1}{1+1} + \frac{1}{4+1} + \frac{1}{9+1} = \frac{9}{5}$

86. $\displaystyle\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

87. $\sum_{i=1}^{4}[(i-1)^2+(i+1)^3] = [(0)^2+(2)^3]+[(1)^2+(3)^3]+[(2)^2+(4)^3]+[(3)^2+(5)^3] = 238$

88. $\sum_{k=2}^{5}(k+1)(k-3) = (2+1)(2-3)+(3+1)(3-3)+(4+1)(4-3)+(5+1)(5-3) = 14$

89. $\sum_{i=1}^{4}2^i = 2^1+2^2+2^3+2^4 = 30$

90. $\sum_{j=0}^{4}(-2)^j = (-2)^0+(-2)^1+(-2)^2+(-2)^3+(-2)^4 = 11$

91. $\sum_{j=1}^{6}(24-3j) = 81$

92. $\sum_{j=1}^{10}\frac{3}{j+1} \approx 6.06$

93. $\sum_{k=0}^{4}\frac{(-1)^k}{k+1} = \frac{47}{60}$

94. $\sum_{k=0}^{4}\frac{(-1)^k}{k!} = \frac{3}{8} = 0.375$

95. $\frac{1}{3(1)}+\frac{1}{3(2)}+\frac{1}{3(3)}+\cdots+\frac{1}{3(9)} = \sum_{i=1}^{9}\frac{1}{3i} \approx 0.94299$

96. $\frac{5}{1+1}+\frac{5}{1+2}+\frac{5}{1+3}+\cdots+\frac{5}{1+15} = \sum_{i=1}^{15}\frac{5}{1+i} \approx 11.904$

97. $\left[2\left(\frac{1}{8}\right)+3\right]+\left[2\left(\frac{2}{8}\right)+3\right]+\left[2\left(\frac{3}{8}\right)+3\right]+\cdots+\left[2\left(\frac{8}{8}\right)+3\right] = \sum_{i=1}^{8}\left[2\left(\frac{i}{8}\right)+3\right] = 33$

98. $\left[1-\left(\frac{1}{6}\right)^2\right]+\left[1-\left(\frac{2}{6}\right)^2\right]+\cdots+\left[1-\left(\frac{6}{6}\right)^2\right] = \sum_{k=1}^{6}\left[1-\left(\frac{k}{6}\right)^2\right] \approx 3.472$

99. $3-9+27-81+243-729 = \sum_{i=1}^{6}(-1)^{i+1}3^i = -546$

100. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots-\frac{1}{128} = \frac{1}{2^0}-\frac{1}{2^1}+\frac{1}{2^2}-\frac{1}{2^3}+\cdots-\frac{1}{2^7} = \sum_{n=0}^{7}\left(-\frac{1}{2}\right)^n \approx 0.664$

101. $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots-\frac{1}{20^2} = \sum_{i=1}^{20}\frac{(-1)^{i+1}}{i^2} \approx 0.82128$

102. $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{10\cdot 12} = \sum_{k=1}^{10}\frac{1}{k(k+2)} \approx 0.663$

103. $\frac{1}{4}+\frac{3}{8}+\frac{7}{16}+\frac{15}{32}+\frac{31}{64} = \sum_{i=1}^{5}\frac{2^i-1}{2^{i+1}} = \frac{129}{64} = 2.015625$

104. $\frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \frac{24}{16} + \frac{120}{32} + \frac{720}{64} = \sum_{k=1}^{6} \frac{k!}{2^k} = 18.25$

105. $\sum_{i=1}^{4} 5\left(\frac{1}{2}\right)^i = 4.6875 = \frac{75}{16}$

106. $\sum_{i=1}^{5} 2\left(\frac{1}{3}\right)^i = \frac{242}{243} \approx 0.9959$

107. $\sum_{n=1}^{3} 4\left(-\frac{1}{2}\right)^n = -1.5 = -\frac{3}{2}$

108. $\sum_{n=1}^{4} 8\left(-\frac{1}{4}\right)^n = \frac{-51}{32} \approx -1.59375$

109. (a) $\sum_{i=1}^{4} 6\left(\frac{1}{10}\right)^i = 6\left(\frac{1}{10}\right) + 6\left(\frac{1}{10}\right)^2 + 6\left(\frac{1}{10}\right)^3 + 6\left(\frac{1}{10}\right)^4$

$= 0.6666$

$= \frac{3333}{5000}$

(b) $\sum_{i=1}^{\infty} 6\left(\frac{1}{10}\right)^i = 6[0.1 + 0.01 + 0.001 + \ldots]$

$= 6[0.111\ldots]$

$= 0.666\ldots$

$= \frac{2}{3}$

110. (a) $\sum_{k=1}^{4} 4\left(\frac{1}{10}\right)^k = 4\left(\frac{1}{10}\right) + 4\left(\frac{1}{10}\right)^2 + 4\left(\frac{1}{10}\right)^3 + 4\left(\frac{1}{10}\right)^4$

$= 0.4444$

$= \frac{1111}{2500}$

(b) $\sum_{k=1}^{\infty} 4\left(\frac{1}{10}\right)^k = 4[0.1 + 0.01 + 0.001 + \ldots]$

$= 4[0.111\ldots]$

$= 0.444\ldots$

$= \frac{4}{9}$

111. (a) $\sum_{k=1}^{4} \left(\frac{1}{10}\right)^k = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10{,}000}$

$= 0.1111$

$= \frac{1111}{10{,}000}$

(b) $\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k = 0.1 + 0.01 + 0.001 + \ldots$

$= 0.111\ldots$

$= \frac{1}{9}$

112. (a) $\sum_{i=1}^{4} 2\left(\frac{1}{10}\right)^i = 2(0.1) + 2(0.01) + 2(0.001) + 2(0.0001)$

$= 2(0.1111)$

$= 0.2222$

$= \frac{1111}{5000}$

(b) $\sum_{i=1}^{\infty} 2\left(\frac{1}{10}\right)^i = 2(0.1) + 2(0.01) + 2(0.001) + \ldots$

$= 0.2222\ldots$

$= \frac{2}{9}$

113. $A_n = 5000\left(1 + \frac{0.03}{4}\right)^n, \quad n = 1, 2, 3, \ldots$

(a) $A_1 = 5000\left(1 + \frac{0.03}{4}\right)^1 = \$5037.50$

$A_2 \approx \$5075.28 \qquad A_3 \approx \$5113.35$

$A_4 \approx \$5151.70 \qquad A_5 \approx \$5190.33$

$A_6 \approx \$5229.26 \qquad A_7 \approx \$5268.48$

$A_8 \approx \$5307.99$

(b) $A_{40} \approx \$6741.74$

114. (a) $A_1 = 100(101)[(1.01)^1 - 1] = \$101.00$

$A_2 = 100(101)[(1.01)^2 - 1] = \$203.01$

$A_3 = 100(101)[(1.01)^3 - 1] \approx \$306.04$

$A_4 = 100(101)[(1.01)^4 - 1] \approx \$410.10$

$A_5 = 100(101)[(1.01)^5 - 1] \approx \$515.20$

$A_6 = 100(101)[(1.01)^6 - 1] \approx \$621.35$

(b) $A_{60} = 100(101)[(1.01)^{60} - 1] \approx \$8248.64$

(c) $A_{240} = 100(101)[(1.01)^{240} - 1] \approx \$99{,}914.79$

115. (a) $p_0 = 5500$ (year 2008)

$p_n = 0.75p_{n-1} + 500$

(b) $p_1 = 0.75p_0 + 500 = 4625$ (2009)

$p_2 = 0.75p_1 + 500 \approx 3969$ (2010)

$p_3 = 0.75p_2 + 500 \approx 3477$ (2011)

(Answers will vary slightly.)

(c) The population approaches 2000 trout because $0.75(2000) + 500 = 2000$.

116. (a) $t_0 = 10{,}000$ (year 2010)

$t_n = 0.9t_{n-1} + 750$

(b) $t_1 = 0.9t_0 + 750 = 9750$

$t_2 = 0.9t_1 + 750 = 9525$

$t_3 \approx 9323$

$t_4 \approx 9140$

(c) The number of trees approaches 7500 because $0.9(7500) + 750 = 7500$.

117. (a) $a_0 = 50$ (end of January)

$a_n = \left(1 + \frac{0.06}{12}\right)a_{n-1} + 50 = 1.005a_{n-1} + 50$

(b) $a_1 = 1.005\,a_0 + 50 = 100.25$ (end of February)

$a_2 = 1.005\,a_1 + 50 = 150.75$

$a_3 = 201.51 \qquad a_4 = 252.51$

$a_5 = 303.78 \qquad a_6 = 355.29$

$a_7 = 407.07 \qquad a_8 = 459.11$

$a_9 = 511.40 \qquad a_{10} = 563.96$

$a_{11} = 616.78$ (end of December)

After one year, the IRA has \$616.78.

(c) After 50 deposits, $a_{49} \approx \$2832.26$.

118. Monthly interest rate is $\frac{0.09}{12} = 0.0075$.

(a) $b_0 = 150{,}000$

$b_1 = b_0(1.0075) - 1206.94 = 149{,}918.06$

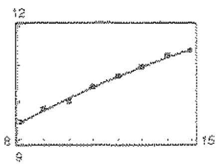
$b_n = b_{n-1}(1.0075) - 1206.94$

(b)

n	0	60	120	180	240	300	360
b_n	150,000	143,819.75	134,143.44	118,993.43	95,273.35	58,135.27	−11.12

(c) Total amount: $1206.94 \times 360 - 11.12 = \$434.487.28$

(d) Total interest: $434{,}487.28 - \$150{,}000 = \$284{,}487.28$

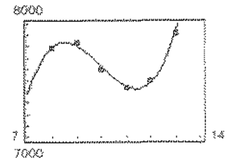
119. (a) 

(b) For 2010, $n = 20$ and $r_{20} \approx \$12.25$.

For 2015, $n = 25$ and $r_{25} \approx \$12.64$.

(c) Answers will vary.

(d) $r_{18} = 11.97$ and $r_{19} = 12.12$. So, the average hourly wage reaches \$12 in 2008.

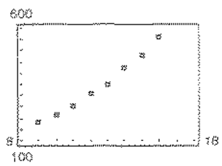
120. (a) 

(b) For 2005, $n = 15$ and $S_{15} \approx 10{,}876$ thousand.

For 2010, $n = 20$ and $S_{20} \approx 39{,}671$ thousand.

For 2015, $n = 25$ and $S_{25} \approx 119{,}349$ thousand.

(c) Answers will vary.

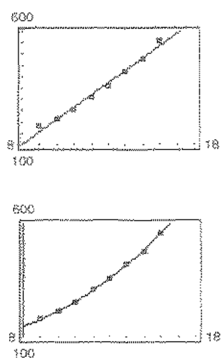
121. (a) 

(b) Linear: $R_n = 54.58n - 336.3$

Quadratic: $R_n = 3.088n^2 - 22.62n + 130.0$

Coefficient of determination for linear model: 0.98656

Coefficient of determination for quadratic model: 0.99919

(c) 

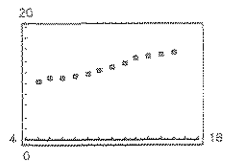
(d) The quadratic model is better.

The quadratic model is better because its coefficient of determination is closer to 1.

(e) For 2010, $n = 20$ and $R_{20} \approx 912.8$ million.

For 2015, $n = 25$ and $R_{25} \approx 1494.5$ million.

(f) $R_n = 1000$ when $n \approx 20.8$, or in 2010.

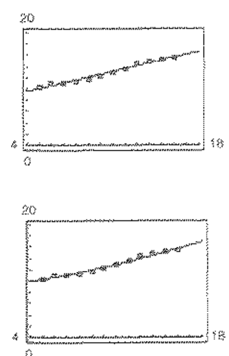
122. (a) 

(b) Linear: $S_n = 0.50n + 7.6$

Quadratic: $S_n = 0.012n^2 + 0.24n + 8.8$

Coefficient of determination for linear model: 0.98201

Coefficient of determination for quadratic model: 0.98759

(c) 

(d) The quadratic model is better.

The quadratic model is better because its coefficient of determination is closer to 1.

(e) For 2010, $n = 20$ and $S_{20} \approx 18.4$ million.

For 2015, $n = 25$ and $S_{25} \approx 22.3$ million.

(f) $S_{22} \approx 19.9$ when $S_{23} \approx 20.7$. So, sales will reach 20 billion in 2012.

123. True

124. True

$$\sum_{j=1}^{4} 2^j = 2^1 + 2^2 + 2^3 + 2^4 = \sum_{j=3}^{6} 2^{j-2}$$

125. $a_0 = 1, a_1 = 1,\ a_{k+2} = a_{k+1} + a_k$

$a_0 = 1$	$b_0 = \frac{1}{1} = 1$	$a_6 = 8 + 5 = 13$	$b_6 = \frac{21}{13}$
$a_1 = 1$	$b_1 = \frac{2}{1} = 2$	$a_7 = 13 + 8 = 21$	$b_7 = \frac{34}{21}$
$a_2 = 1 + 1 = 2$	$b_2 = \frac{3}{2}$	$a_8 = 21 + 13 = 34$	$b_8 = \frac{55}{34}$
$a_3 = 2 + 1 = 3$	$b_3 = \frac{5}{3}$	$a_9 = 34 + 21 = 55$	$b_9 = \frac{89}{55}$
$a_4 = 3 + 2 = 5$	$b_4 = \frac{8}{5}$	$a_{10} = 55 + 34 = 89$	
$a_5 = 5 + 3 = 8$	$b_5 = \frac{13}{8}$	$a_{11} = 89 + 55 = 144$	

126. $$b_n = \frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n}$$
$$= 1 + \frac{a_{n-1}}{a_n} = 1 + \frac{1}{\frac{a_n}{a_{n-1}}} = 1 + \frac{1}{b_{n-1}}$$

127. $$a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$
$$a_1 = \frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2^1\sqrt{5}} = 1$$
$a_2 = 1, \quad a_3 = 2$

$a_4 = 3, \quad a_5 = 5$

128. These are the first five terms of the Fibonacci sequence.

129. $$a_{n+1} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}$$
$$a_{n+2} = \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}}$$

130. $$a_{n+1} + a_n = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} + \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$
$$= \frac{2(1+\sqrt{5})^{n+1} - 2(1-\sqrt{5})^{n+1} + 4(1+\sqrt{5})^n - 4(1-\sqrt{5})^n}{2^{n+2}\sqrt{5}}$$
$$= \frac{(1+\sqrt{5})^n[2(1+\sqrt{5}) + 4] - (1-\sqrt{5})^n[2(1-\sqrt{5}) + 4]}{2^{n+2}\sqrt{5}}$$
$$= \frac{(1+\sqrt{5})^n(1+\sqrt{5})^2 - (1-\sqrt{5})^n(1-\sqrt{5})^2}{2^{n+2}\sqrt{5}}$$
$$= \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}}$$
$$= a_{n+2}$$

Yes, this is the recursive formula for the Fibonacci sequence.

131. $a_n = \dfrac{x^n}{n!}$

$a_1 = \dfrac{x}{1} = x$

$a_2 = \dfrac{x^2}{2!} = \dfrac{x^2}{2}$

$a_3 = \dfrac{x^3}{3!} = \dfrac{x^3}{6}$

$a_4 = \dfrac{x^4}{4!} = \dfrac{x^4}{24}$

$a_5 = \dfrac{x^5}{5!} = \dfrac{x^5}{120}$

132. $a_n = \dfrac{x^2}{n^2}$

$a_1 = \dfrac{x^2}{1}$

$a_2 = \dfrac{x^2}{4}$

$a_3 = \dfrac{x^2}{9}$

$a_4 = \dfrac{x^2}{16}$

$a_5 = \dfrac{x^2}{25}$

133. $a_n = \dfrac{(-1)^n x^{2n+1}}{2n+1}$

$a_1 = \dfrac{-x^3}{3}$

$a_2 = \dfrac{x^5}{5}$

$a_3 = -\dfrac{x^7}{7}$

$a_4 = \dfrac{x^9}{9}$

$a_5 = \dfrac{-x^{11}}{11}$

134. $a_n = \dfrac{(-1)^n x^{n+1}}{n+1}$

$a_1 = \dfrac{-x^2}{2}$

$a_2 = \dfrac{x^3}{3}$

$a_3 = \dfrac{-x^4}{4}$

$a_4 = \dfrac{x^5}{5}$

$a_5 = \dfrac{-x^6}{6}$

135. $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$

$a_1 = \dfrac{-x^2}{2}$

$a_2 = \dfrac{x^4}{4!} = \dfrac{x^4}{24}$

$a_3 = \dfrac{-x^6}{6!} = \dfrac{-x^6}{720}$

$a_4 = \dfrac{x^8}{8!} = \dfrac{x^8}{40{,}320}$

$a_5 = \dfrac{-x^{10}}{10!} = \dfrac{-x^{10}}{3{,}628{,}800}$

136. $a_n = \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$

$a_1 = \dfrac{-x^3}{3!} = \dfrac{-x^3}{6}$

$a_2 = \dfrac{x^5}{5!} = \dfrac{x^5}{120}$

$a_3 = -\dfrac{x^7}{7!} = \dfrac{-x^7}{5040}$

$a_4 = \dfrac{x^9}{9!} = \dfrac{x^9}{362{,}880}$

$a_5 = \dfrac{-x^{11}}{11!} = \dfrac{-x^{11}}{39{,}916{,}800}$

137. $a_n = \dfrac{(-1)^n x^n}{n!}$

$a_1 = -x$

$a_2 = \dfrac{x^2}{2}$

$a_3 = \dfrac{-x^3}{3!} = \dfrac{-x^3}{6}$

$a_4 = \dfrac{x^4}{4!} = \dfrac{x^4}{24}$

$a_5 = \dfrac{-x^5}{5!} = -\dfrac{x^5}{120}$

138. $a_n = \dfrac{(-1)^n x^{n+1}}{(n+1)!}$

$a_1 = \dfrac{-x^2}{2}$

$a_2 = \dfrac{x^3}{3!} = \dfrac{x^3}{6}$

$a_3 = \dfrac{-x^4}{4!} = -\dfrac{x^4}{24}$

$a_4 = \dfrac{x^5}{5!} = \dfrac{x^5}{120}$

$a_5 = \dfrac{-x^6}{6!} = -\dfrac{x^6}{720}$

139. $a_n = \dfrac{(-1)^{n+1}(x+1)^n}{n!}$

$a_1 = x + 1$

$a_2 = \dfrac{-(x+1)^2}{2}$

$a_3 = \dfrac{(x+1)^3}{6}$

$a_4 = -\dfrac{(x+1)^4}{24}$

$a_5 = \dfrac{(x+1)^5}{120}$

140. $a_n = \dfrac{(-1)^n (x-1)^n}{(n+1)!}$

$a_1 = \dfrac{-(x-1)}{2}$

$a_2 = \dfrac{(x-1)^2}{6}$

$a_3 = \dfrac{-(x-1)^3}{24}$

$a_4 = \dfrac{(x-1)^4}{120}$

$a_5 = \dfrac{-(x-1)^5}{720}$

141. $a_n = \dfrac{1}{2n} - \dfrac{1}{2n+2}$

$a_1 = \dfrac{1}{2} - \dfrac{1}{4} = \dfrac{1}{4}$

$a_2 = \dfrac{1}{4} - \dfrac{1}{6} = \dfrac{1}{12}$

$a_3 = \dfrac{1}{6} - \dfrac{1}{8} = \dfrac{1}{24}$

$a_4 = \dfrac{1}{8} - \dfrac{1}{10} = \dfrac{1}{40}$

$a_5 = \dfrac{1}{10} - \dfrac{1}{12} = \dfrac{1}{60}$

nth partial sum $= \left(\dfrac{1}{2} - \dfrac{1}{4}\right) + \left(\dfrac{1}{4} - \dfrac{1}{6}\right) + \cdots + \left(\dfrac{1}{2n} - \dfrac{1}{2n+2}\right) = \dfrac{1}{2} - \dfrac{1}{2n+2}$

142. $a_n = \dfrac{1}{n} - \dfrac{1}{n+1}$

$a_1 = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

$a_2 = \dfrac{1}{2} - \dfrac{1}{3} = \dfrac{1}{6}$

$a_3 = \dfrac{1}{3} - \dfrac{1}{4} = \dfrac{1}{12}$

$a_4 = \dfrac{1}{4} - \dfrac{1}{5} = \dfrac{1}{20}$

$a_5 = \dfrac{1}{5} - \dfrac{1}{6} = \dfrac{1}{30}$

nth partial sum $= \left(\dfrac{1}{1} - \dfrac{1}{2}\right) + \left(\dfrac{1}{2} - \dfrac{1}{3}\right) + \cdots + \left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) = 1 - \dfrac{1}{n+1}$

143. $a_n = \dfrac{1}{n+1} - \dfrac{1}{n+2}$

$a_1 = \dfrac{1}{2} - \dfrac{1}{3} = \dfrac{1}{6}$

$a_2 = \dfrac{1}{3} - \dfrac{1}{4} = \dfrac{1}{12}$

$a_3 = \dfrac{1}{4} - \dfrac{1}{5} = \dfrac{1}{20}$

$a_4 = \dfrac{1}{5} - \dfrac{1}{6} = \dfrac{1}{30}$

$a_5 = \dfrac{1}{6} - \dfrac{1}{7} = \dfrac{1}{42}$

nth partial sum $= \left(\dfrac{1}{2} - \dfrac{1}{3}\right) + \left(\dfrac{1}{3} - \dfrac{1}{4}\right) + \cdots + \left(\dfrac{1}{n+1} - \dfrac{1}{n+2}\right) = \dfrac{1}{2} - \dfrac{1}{n+2}$

144. $a_n = \dfrac{1}{n} - \dfrac{1}{n+2}$

$a_1 = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$a_2 = \dfrac{1}{2} - \dfrac{1}{4} = \dfrac{1}{4}$

$a_3 = \dfrac{1}{3} - \dfrac{1}{5} = \dfrac{2}{15}$

$a_4 = \dfrac{1}{4} - \dfrac{1}{6} = \dfrac{1}{12}$

$a_5 = \dfrac{1}{5} - \dfrac{1}{7} = \dfrac{2}{35}$

nth partial sum $= \left(1 - \dfrac{1}{3}\right) + \left(\dfrac{1}{2} - \dfrac{1}{4}\right) + \left(\dfrac{1}{3} - \dfrac{1}{5}\right) + \cdots + \left(\dfrac{1}{n-1} - \dfrac{1}{n+1}\right) + \left(\dfrac{1}{n} - \dfrac{1}{n+2}\right)$

$= \left(1 + \dfrac{1}{2}\right) - \left(\dfrac{1}{n+1} + \dfrac{1}{n+2}\right) = \dfrac{3}{2} - \dfrac{1}{n+1} - \dfrac{1}{n+2}$

145. $a_n = \ln n$

$a_1 = \ln 1 = 0$

$a_2 = \ln 2$

$a_3 = \ln 3$

$a_4 = \ln 4$

$a_5 = \ln 5$

nth partial sum $= \ln 2 + \ln 3 + \cdots + \ln n$

$= \ln(2 \cdot 3 \cdots n)$

$= \ln(n!)$

146. $a_n = 1 - \ln(n + 1)$

$a_1 = 1 - \ln 2$

$a_2 = 1 - \ln 3$

$a_3 = 1 - \ln 4$

$a_4 = 1 - \ln 5$

$a_5 = 1 - \ln 6$

$$\begin{aligned} n\text{th partial sum} &= (1 - \ln 2) + (1 - \ln 3) + \cdots + (1 - \ln(n + 1)) \\ &= n - [\ln 2 + \ln 3 + \cdots + \ln(n + 1)] \\ &= n - \ln(2 \cdot 3 \cdots (n + 1)) \\ &= n - \ln((n + 1)!) \end{aligned}$$

147. (a) $A - B = \begin{bmatrix} 8 & 1 \\ -3 & 7 \end{bmatrix}$

(b) $2B - 3A = \begin{bmatrix} -22 & -7 \\ 3 & -18 \end{bmatrix}$

(c) $AB = \begin{bmatrix} 18 & 9 \\ 18 & 0 \end{bmatrix}$

(d) $BA = \begin{bmatrix} 0 & 6 \\ 27 & 18 \end{bmatrix}$

148. (a) $A - B = \begin{bmatrix} 10 & 19 \\ -12 & -5 \end{bmatrix}$

(b) $2B - 3A = \begin{bmatrix} -30 & -45 \\ 28 & 4 \end{bmatrix}$

(c) $AB = \begin{bmatrix} 56 & -43 \\ 48 & 114 \end{bmatrix}$

(d) $BA = \begin{bmatrix} 48 & -72 \\ 36 & 122 \end{bmatrix}$

149. (a) $A - B = \begin{bmatrix} -3 & -7 & 4 \\ 4 & 4 & 1 \\ 1 & 4 & 3 \end{bmatrix}$

(b) $2B - 3A = \begin{bmatrix} 8 & 17 & -14 \\ -12 & -13 & -9 \\ -3 & -15 & -10 \end{bmatrix}$

(c) $AB = \begin{bmatrix} -2 & 7 & -16 \\ 4 & 42 & 45 \\ 1 & 23 & 48 \end{bmatrix}$

(d) $BA = \begin{bmatrix} 16 & 31 & 42 \\ 10 & 47 & 31 \\ 13 & 22 & 25 \end{bmatrix}$

150. (a) $A - B = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & 4 \\ 1 & -1 & 1 \end{bmatrix}$

(b) $2B - 3A = \begin{bmatrix} 3 & -4 & 0 \\ -9 & -1 & -10 \\ -2 & 3 & -5 \end{bmatrix}$

(c) $AB = \begin{bmatrix} 12 & 0 & -8 \\ 1 & 21 & 2 \\ -6 & -1 & 8 \end{bmatrix}$

(d) $BA = \begin{bmatrix} 20 & 4 & 8 \\ 2 & 15 & -4 \\ 1 & -6 & 6 \end{bmatrix}$

Section 8.2 Arithmetic Sequences and Partial Sums

- You should be able to recognize an arithmetic sequence, find its common difference, and find its nth term.
- You should be able to find the nth partial sum of an arithmetic sequence with common difference d using the formula

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Vocabulary Check

1. arithmetic, common

2. $a_n = dn + c$

3. nth partial sum

1. $10, 8, 6, 4, 2, \ldots$

Arithmetic sequence, $d = -2$

2. $4, 9, 14, 19, 24, \ldots$

Arithmetic sequence, $d = 5$

3. $3, \frac{5}{2}, 2, \frac{3}{2}, 1, \ldots$

Arithmetic sequence, $d = -\frac{1}{2}$

4. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \ldots$

Not an arithmetic sequence

5. $-24, -16, -8, 0, 8$

Arithmetic sequence, $d = 8$

6. $\ln 1, \ln 2, \ln 3, \ln 4, \ln 5, \ldots$

Not an arithmetic sequence

7. $3.7, 4.3, 4.9, 5.5, 6.1, \ldots$

Arithmetic sequence, $d = 0.6$

8. $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

Not an arithmetic sequence

9. $a_n = 8 + 13n$

$21, 34, 47, 60, 73$

Arithmetic sequence, $d = 13$

10. $a_n = 2^n + n$

$3, 6, 11, 20, 37$

Not an arithmetic sequence

11. $a_n = \dfrac{1}{n+1}$

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$

Not an arithmetic sequence

12. $a_n = 1 + (n - 1)4$

$1, 5, 9, 13, 17$

Arithmetic sequence, $d = 4$

13. $a_n = 150 - 7n$

$143, 136, 129, 122, 115$

Arithmetic sequence, $d = -7$

14. $a_n = 2^{n-1}$

$1, 2, 4, 8, 16$

Not an arithmetic sequence

15. $a_n = 3 + 2(-1)^n$

$1, 5, 1, 5, 1$

Not an arithmetic sequence

16. $a_n = 3 - 4(n + 6) = -21 - 4n$

$a_1 = -25$

$a_2 = -29$

$a_3 = -33$

$a_4 = -37$

$a_5 = -41$

Arithmetic sequence, $d = -4$

17. $a_1 = 1, d = 3$

$a_n = a_1 + (n - 1)d = 1 + (n - 1)(3) = 3n - 2$

18. $a_1 = 15, d = 4$

$a_n = a_1 + (n - 1)d = 15 + (n - 1)4 = 11 + 4n$

19. $a_1 = 100, d = -8$

$a_n = a_1 + (n - 1)d$

$= 100 + (n - 1)(-8) = 108 - 8n$

20. $a_1 = 0, d = -\frac{2}{3}$

$a_n = a_1 + (n - 1)d = (n - 1)\left(-\frac{2}{3}\right) = \frac{2}{3} - \frac{2}{3}n$

21. $4, \frac{3}{2}, -1, -\frac{7}{2}, \ldots, d = -\frac{5}{2}$

$a_n = a_1 + (n - 1)d = 4 + (n - 1)\left(-\frac{5}{2}\right) = \frac{13}{2} - \frac{5}{2}n$

22. $10, 5, 0, -5, -10, \ldots, d = -5$

$a_n = a_1 + (n - 1)d = 10 + (n - 1)(-5)$

$= 15 - 5n$

23. $a_1 = 5, a_4 = 15$

$a_4 = a_1 + 3d \implies 15 = 5 + 3d \implies d = \frac{10}{3}$

$a_n = a_1 + (n - 1)d = 5 + (n - 1)\left(\frac{10}{3}\right) = \frac{10}{3}n + \frac{5}{3}$

24. $a_1 = -4, a_5 = 16$

$a_n = a_1 + (n - 1)d$

$16 = -4 + 4d$

$d = 5$

$a_n = -4 + (n - 1)5 = -9 + 5n$

25. $a_3 = 94,\ a_6 = 85$

$a_6 = a_3 + 3d \implies 85 = 94 + 3d \implies d = -3$

$a_1 = a_3 - 2d \implies a_1 = 94 - 2(-3) = 100$

$a_n = a_1 + (n - 1)d$

$= 100 + (n - 1)(-3) = 103 - 3n$

26. $a_5 = 190, a_{10} = 115$

$a_{10} = a_5 + 5d \implies 115 = 190 + 5d \implies d = -15$

$a_1 = a_5 - 4d \implies a_1 = 190 - 4(-15) = 250$

$a_n = a_1 + (n - 1)d = 250 + (n - 1)(-15)$

$= 265 - 15n$

27. $a_1 = 5,\ d = 6$

$a_1 = 5$

$a_2 = 5 + 6 = 11$

$a_3 = 11 + 6 = 17$

$a_4 = 17 + 6 = 23$

$a_5 = 23 + 6 = 29$

28. $a_1 = 5, d = -\frac{3}{4}$

$a_1 = 5$

$a_2 = 5 - \frac{3}{4} = \frac{17}{4}$

$a_3 = \frac{17}{4} - \frac{3}{4} = \frac{14}{4} = \frac{7}{2}$

$a_4 = \frac{7}{2} - \frac{3}{4} = \frac{11}{4}$

$a_5 = \frac{11}{4} - \frac{3}{4} = \frac{8}{4} = 2$

29. $a_1 = -10,\quad d = -12$

$a_1 = -10$

$a_2 = -10 - 12 = -22$

$a_3 = -22 - 12 = -34$

$a_4 = -34 - 12 = -46$

$a_5 = -46 - 12 = -58$

30. $a_4 = 16, a_{10} = 46$

$16 = a_4 = a_1 + (n - 1)d = a_1 + 3d$

$46 = a_{10} = a_1 + (n - 1)d = a_1 + 9d$

Answer: $a_1 = 1, d = 5$

$a_1 = 1$

$a_2 = 1 + 5 = 6$

$a_3 = 6 + 5 = 11$

$a_4 = 11 + 5 = 16$

$a_5 = 16 + 5 = 21$

31. $a_8 = 26,\ a_{12} = 42$

$26 = a_8 = a_1 + (n - 1)d = a_1 + 7d$

$42 = a_{12} = a_1 + (n - 1)d = a_1 + 11d$

Answer: $d = 4,\ a_1 = -2$

$a_1 = -2$

$a_2 = -2 + 4 = 2$

$a_3 = 2 + 4 = 6$

$a_4 = 6 + 4 = 10$

$a_5 = 10 + 4 = 14$

32. $a_{11} = a_6 + 5d$

$-73 = -38 + 5d \implies d = -7$

$a_6 = a_1 + 5d \implies -38 = a_1 + 5(-7) \implies a_1 = -3$

$a_2 = -3 - 7 = -10$

$a_3 = -10 - 7 = -17$

$a_4 = -17 - 7 = -24$

$a_5 = -24 - 7 = -31$

33. $a_3 = 19, a_{15} = -1.7$

$a_{15} = a_3 + 12d$

$-1.7 = 19 + 12d \implies d = -1.725$

$a_3 = a_1 + 2d \implies 19 = a_1 + 2(-1.725)$

$\implies a_1 = 22.45$

$a_2 = a_1 - 1.725 = 20.725$

$a_3 = 19$

$a_4 = 19 - 1.725 = 17.275$

$a_5 = 17.275 - 1.725 = 15.55$

34. $a_{14} = a_5 + 9d$

$38.5 = 16 + 9d \implies d = 2.5$

$a_5 = a_1 + 4d \implies 16 = a_1 + 4(2.5) \implies a_1 = 6$

$a_2 = 6 + 2.5 = 8.5$

$a_3 = 8.5 + 2.5 = 11$

$a_4 = 11 + 2.5 = 13.5$

$a_5 = 13.5 + 2.5 = 16$

35. $a_1 = 15, \quad a_{k+1} = a_k + 4$

$a_2 = a_1 + 4 = 15 + 4 = 19$

$a_3 = 19 + 4 = 23$

$a_4 = 23 + 4 = 27$

$a_5 = 27 + 4 = 31$

$d = 4, \quad a_n = 11 + 4n$

36. $a_1 = 200$

$a_{k+1} = a_k - 10$

$a_2 = 200 - 10 = 190$

$a_3 = 190 - 10 = 180$

$a_4 = 180 - 10 = 170$

$a_5 = 170 - 10 = 160$

$d = -10$

$a_n = -10n + 210$

37. $a_1 = \frac{3}{5}, a_{k+1} = -\frac{1}{10} + a_k$

$a_2 = -\frac{1}{10} + \frac{3}{5} = \frac{5}{10} = \frac{1}{2}$

$a_3 = -\frac{1}{10} + \frac{1}{2} = \frac{4}{10} = \frac{2}{5}$

$a_4 = -\frac{1}{10} + \frac{2}{5} = \frac{3}{10}$

$a_5 = -\frac{1}{10} + \frac{3}{10} = \frac{1}{5}$

$d = -\frac{1}{10}$

$a_n = \frac{7}{10} - \frac{1}{10}n$

38. $a_1 = 1.5, a_{k+1} = a_k - 2.5$

$a_2 = 1.5 - 2.5 = -1.0$

$a_3 = -1.0 - 2.5 = -3.5$

$a_4 = -3.5 - 2.5 = -6.0$

$a_5 = -6.0 - 2.5 = -8.5$

$d = -2.5$

$a_n = 4.0 - 2.5n$

39. $a_1 = 5, a_2 = 11 \implies d = 6$

$a_{10} = a_1 + 9d = 5 + 9(6) = 59$

40. $a_2 = a_1 + d$

$13 = 3 + d \implies d = 10$

$a_9 = a_1 + 8d$

$= 3 + 8(10) = 83$

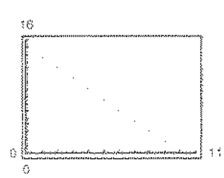
41. $a_1 = 4.2, a_2 = 6.6 \implies d = 2.4$

$a_7 = a_1 + 6d = 4.2 + 6(2.4) = 18.6$

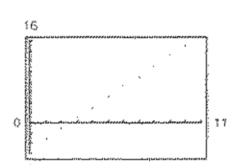
42. $d = a_2 - a_1 = -13.8 - (-0.7) = -13.1$

$a_8 = a_1 + 7d = -0.7 + 7(-13.1) = -92.4$

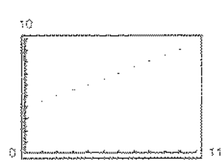
43. $a_n = 15 - \frac{3}{2}n$



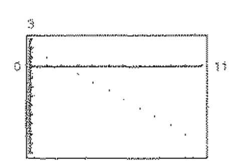
44. $a_n = -5 + 2n$



45. $a_n = 0.5n + 4$



46. $a_n = -0.9n + 2$



47. $a_n = 4n - 5$

n	1	2	3	4	5	6	7	8	9	10
a_n	-1	3	7	11	15	19	23	27	31	35

48. $a_n = 17 + 3n$

n	1	2	3	4	5	6	7	8	9	10
a_n	20	23	26	29	32	35	38	41	44	47

49. $a_n = 20 - \frac{3}{4}n$

n	1	2	3	4	5	6	7	8	9	10
a_n	19.25	18.5	17.75	17	16.25	15.5	14.75	14	13.25	12.5

50. $a_n = \frac{4}{5}n + 12$

n	1	2	3	4	5	6	7	8	9	10
a_n	12.8	13.6	14.4	15.2	16	16.8	17.6	18.4	19.2	20

51. $a_n = 1.5 + 0.05n$

n	1	2	3	4	5	6	7	8	9	10
a_n	1.55	1.6	1.65	1.7	1.75	1.8	1.85	1.9	1.95	2.0

52. $a_n = 8 - 12.5n$

n	1	2	3	4	5	6	7	8	9	10
a_n	-4.5	-17	-29.5	-42	-54.5	-67	-79.5	-92	-104.5	-117

53. $S_{10} = \frac{10}{2}(2 + 20) = 110$

54. $S_7 = \frac{7}{2}(1 + 19) = 70$

55. $S_5 = \frac{5}{2}(-1 + (-9)) = -25$

56. $S_6 = \frac{6}{2}(-5 + 5) = 0$

57. $S_{50} = \frac{50}{2}(2 + 100) = 2550$

58. $a_1 = 1,\ a_{100} = 199,\ n = 100$

$$\sum_{n=1}^{100}(2n - 1) = \frac{100}{2}(1 + 199)$$

$$= 10{,}000$$

59. $S_{131} = \frac{131}{2}(-100 + 30) = -4585$

60. $a_1 = -10, a_{61} = 50, n = 61$

$$\sum_{i=0}^{60}(i - 10) = \frac{61}{2}(-10 + 50) = 1220$$

61. $8, 20, 32, 44, \ldots \; n = 10$

$a_1 = 8, a_2 = 20 \implies d = 12$

$a_{10} = a_1 + 9d = 8 + 9(12) = 116$

$S_{10} = \frac{n}{2}[a_1 + a_{10}] = \frac{10}{2}[8 + 116] = 620$

62. $-6, -2, 2, 6, \ldots$

$a_1 = -6, d = 4, n = 50$

$a_{50} = -6 + 49(4) = 190$

$S_{50} = \frac{50}{2}(-6 + 190) = 4600$

63. $a_1 = 0.5, a_2 = 1.3 \implies d = 0.8$

$a_{10} = a_1 + 9d = 0.5 + 9(0.8) = 7.7$

$S_{10} = \frac{10}{2}(a_1 + a_{10}) = 5(0.5 + 7.7) = 41$

64. $4.2, 3.7, 3.2, 2.7, \ldots \; n = 12$

$a_1 = 4.2$

$d = -0.5$

$n = 12$

$a_{12} = 4.2 + 11(-0.5) = -1.3$

$S_{12} = \frac{12}{2}(4.2 - 1.3) = 17.4$

65. $a_1 = 100, a_{25} = 220$

$S_{25} = \frac{25}{2}(a_1 + a_{25}) = 12.5(100 + 220) = 4000$

66. $a_1 = 15, a_{100} = 307, n = 100$

$S_{100} = \frac{100}{2}(15 + 307) = 16{,}100$

67. $a_1 = 1, \; a_{50} = 50, \; n = 50$

$\sum_{n=1}^{50} n = \frac{50}{2}(1 + 50) = 1275$

68. $a_n = 2n$

$a_1 = 2, a_{100} = 200, n = 100$

$\sum_{n=1}^{100} 2n = \frac{100}{2}(2 + 200) = 10{,}100$

69. $a_1 = 5, \; a_{100} = 500, n = 100$

$\sum_{n=1}^{100} 5n = \frac{100}{2}(5 + 500) = 25{,}250$

70. $a_n = 7n$

$a_{51} = 357, a_{100} = 700$

$\sum_{n=51}^{100} 7n = \frac{50}{2}(357 + 700) = 26{,}425$

71. $\sum_{n=11}^{30} n - \sum_{n=1}^{10} n = \frac{20}{2}(11 + 30) - \frac{10}{2}(1 + 10)$

$= 410 - 55 = 355$

72. $\sum_{n=51}^{100} n - \sum_{n=1}^{50} n = \frac{50}{2}(51 + 100) - \frac{50}{2}(1 + 50)$

$= 3775 - 1275 = 2500$

73. $\sum_{n=1}^{500} (n + 8) = \frac{500}{2}[9 + 508] = 129{,}250$

74. $a_n = 1000 - n$

$a_1 = 999, a_{250} = 750, n = 250$

$\sum_{n=1}^{250} (1000 - n) = \frac{250}{2}(999 + 750) = 218{,}625$

75. $\sum_{n=1}^{20} (2n + 1) = 440$

76. $\sum_{n=0}^{50} (50 - 2n) = 0$

77. $\sum_{n=1}^{100} \frac{n + 1}{2} = 2575$

78. $\sum_{n=0}^{100} \frac{4 - n}{4} = -1161.5$

79. $\sum_{i=1}^{60} \left(250 - \frac{2}{5}i\right) = 14{,}268$

80. $\sum_{j=1}^{200} (10.5 + 0.025j) = 2602.5$

81. $a_1 = 14, a_{18} = 31$

$S_{18} = \frac{18}{2}(14 + 31) = 405$ bricks

82. $a_1 = 15, a_{10} = 24, d = 1, n = 10$

$S_{10} = \frac{10}{2}(15 + 24) = 195$ logs

83. $a_1 = 20{,}000$

$a_2 = 20{,}000 + 5000 = 25{,}000$

$d = 5000$

$a_5 = 20{,}000 + 4(5000) = 40{,}000$

$S_5 = \frac{5}{2}(20{,}000 + 40{,}000) = 150{,}000$

84. $a_1 = 4.9, a_2 = 14.7, a_3 = 24.5,$

$a_4 = 34.3 \implies d = 9.8$

$a_1 = 4.9 = 9.8(1) + c \implies c = -4.9$

$a_n = 9.8n - 4.9$

$a_{10} = 9.8(10) - 4.9 = 93.1$

$S_{10} = \frac{10}{2}(4.9 + 93.1) = 490$ meters

85. (a) $S_n = 0.91n + 5.7$

(b)

Year	1997	1998	1999	2000	2001	2002	2003	2004
Sales (Billions of \$)	12.1	13.0	13.9	14.8	15.7	16.6	17.5	18.4

The model is a good fit.

(c) Total $= \frac{8}{2}(12.1 + 18.4) = \$122$ billion

(d) For 2005, $n = 15$ and $S_{15} = 19.35$.

For 2012, $n = 22$ and $S_{22} = 25.72$.

Total $= \frac{8}{2}(19.35 + 25.72) = \$180.3$ billion

Answers will vary.

86. (a) $a_n = 13.5n + 324$, $n = 5$ corresponds to 1995.

(b)

Year	Model
1995	392
1996	405
1997	419
1998	432
1999	446
2000	459
2001	473
2002	486
2003	500

(c) $S = \frac{9}{2}[392 + 500] \approx 4014$ thousand

Adding the table entries,

$398 + \cdots + 512 = 4012$ thousand.

(d) For 2004 to 2014,

$S = \frac{11}{2}(513 + 648) \approx 6386$ thousand.

Answers will vary.

87. True. Given a_1 and a_2, you know $d = a_2 - a_1$. Thus, $a_n = a_1 + (n - 1)d$.

88. False. You need to know how many terms are in the sequence.

89. $a_1 = x$ $\quad a_6 = 11x$

$a_2 = x + 2x = 3x$ $\quad a_7 = 13x$

$a_3 = 3x + 2x = 5x$ $\quad a_8 = 15x$

$a_4 = 7x$ $\quad a_9 = 17x$

$a_5 = 9x$ $\quad a_{10} = 19x$

90. $a_1 = -y$ $\quad a_6 = 24y$

$a_2 = -y + 5y = 4y$ $\quad a_7 = 29y$

$a_3 = 9y$ $\quad a_8 = 34y$

$a_4 = 14y$ $\quad a_9 = 39y$

$a_5 = 19y$ $\quad a_{10} = 44y$

91. $a_{20} = a_1 + 19(3) = a_1 + 57$

$$S = \frac{n}{2}(a_1 + a_{20})$$

$$= \frac{20}{2}(a_1 + (a_1 + 57)) = 650$$

$$10(2a_1 + 57) = 650$$

$$20a_1 = 80$$

$$a_1 = 4$$

92. $S = \frac{n}{2}((a_1 + 5) + (a_n + 5))$

$$= \frac{n}{2}(a_1 + a_2 + 10)$$

$$= \frac{n}{2}(a_1 + a_2) + 5n$$

93. (a) $-7, -4, -1, 2, 5, 8, 11$

$a_{n+1} = a_n + 3,\ a_1 = -7$

(b) $17, 23, 29, 35, 41, 47, 53, 57$

$a_{n+1} = a_n + 6,\ a_1 = 17$

(c) Not arithmetic

(d) $4, 7.5, 11, 14.5, 18, 21.5, 25, 28.5$

$a_{n+1} = a_n + 3.5,\ a_1 = 4$

(e) Not arithmetic

94. Gauss might have done the following:

$1 + 2 + 3 + \cdots + 99 + 100 = x$

$100 + 99 + \cdots + 2 + 1 = x$

Adding: $101 + 101 + \cdots + 101 + 101 = 2x$

$100(101) = 2x \implies x = \frac{100(101)}{2} = 5050$

In general, $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.

95. $S = \frac{n(n+1)}{2} = \frac{200(201)}{2} = 20{,}100$

96. $S = 2 + 4 + 6 + \cdots + 200$

$$= 2(1 + 2 + \cdots + 100)$$

$$= 2\left(\frac{100(101)}{2}\right) = 10{,}100$$

97. $S = 1 + 3 + 5 + \cdots + 101$

$$= (1 + 2 + 3 + \cdots + 101) - (2 + 4 + \cdots + 100)$$

$$= \frac{101(102)}{2} - 2\left(\frac{50(51)}{2}\right)$$

$$= 5151 - 2550 = 2601$$

98. $4 + 8 + \cdots + 400 = 4(1 + 2 + \cdots + 100)$

$$= 4 \cdot \frac{100(101)}{2}$$

$$= 200(101) = 20{,}200$$

99. $\begin{bmatrix} 2 & -1 & 7 & \vdots & -10 \\ 3 & 2 & -4 & \vdots & 17 \\ 6 & -5 & 1 & \vdots & -20 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & -1 \end{bmatrix}$.

Answer: $(1, 5, -1)$

100. $\begin{bmatrix} -1 & 4 & 10 & \vdots & 4 \\ 5 & -3 & 1 & \vdots & 31 \\ 8 & 2 & -3 & \vdots & -5 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & -6 \\ 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$.

Answer: $(2, -6, 3)$

101. $\begin{vmatrix} 0 & 0 & 1 \\ 4 & -3 & 1 \\ 2 & 6 & 1 \end{vmatrix} = 30$

Area $= \frac{1}{2}(30) = 15$ square units

102. $\begin{vmatrix} -1 & 2 & 1 \\ 5 & 1 & 1 \\ 3 & 8 & 1 \end{vmatrix} = 40$

Area $= \frac{1}{2}(40) = 20$ square units

103. Answers will vary.

Section 8.3 Geometric Sequences and Series

- You should be able to identify a geometric sequence, find its common ratio, and find the nth term.
- You should be able to find the nth partial sum of a geometric sequence with common ratio r using the formula.

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$$

- You should know that if $|r| < 1$, then

$$\sum_{n=1}^{\infty} a_1 r^{n-1} = \frac{a_1}{1 - r}.$$

Vocabulary Check

1. geometric, common

2. $a_n = a_1 r^{n-1}$

3. $S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1 - r^n}{1 - r}\right)$

4. geometric series

5. $S = \sum_{i=0}^{\infty} a_1 r^i = \dfrac{a_1}{1 - r}$

1. $5, 15, 45, 135, \ldots$

Geometric sequence

$r = 3$

2. $3, 12, 48, 192, \ldots$

Geometric sequence

$r = \frac{12}{3} = 4$

3. $6, 18, 30, 42, \ldots$

Not a geometric sequence

(**Note:** It is an arithmetic sequence with $d = 12$.)

4. $1, -2, 4, -8, \ldots$

Geometric sequence

$r = -2$

5. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

Geometric sequence

$r = -\frac{1}{2}$

6. $5, 1, 0.2, 0.04$

Geometric sequence

$r = \frac{1}{5} = 0.2$

7. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \ldots$

Geometric sequence

$r = 2$

8. $9, -6, 4, -\frac{8}{3}, \ldots$

Geometric sequence

$r = -\frac{2}{3}$

9. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Not a geometric sequence

10. $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \frac{4}{11}, \ldots$

Not a geometric sequence

11. $a_1 = 6, r = 3$

$a_2 = 6(3) = 18$

$a_3 = 18(3) = 54$

$a_4 = 54(3) = 162$

$a_5 = 162(3) = 486$

12. $a_1 = 4, r = 2$

$a_2 = 4(2) = 8$

$a_3 = 8(2) = 16$

$a_4 = 16(2) = 32$

$a_5 = 32(2) = 64$

13. $a_1 = 1,\ r = \frac{1}{2}$

$a_1 = 1$

$a_2 = 1\left(\frac{1}{2}\right) = \frac{1}{2}$

$a_3 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

$a_4 = \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8}$

$a_5 = \frac{1}{8}\left(\frac{1}{2}\right) = \frac{1}{16}$

14. $a_1 = 2, r = \frac{1}{3}$

$a_2 = 2\left(\frac{1}{3}\right) = \frac{2}{3}$

$a_3 = \frac{2}{3}\left(\frac{1}{3}\right) = \frac{2}{9}$

$a_4 = \frac{2}{9}\left(\frac{1}{3}\right) = \frac{2}{27}$

$a_5 = \frac{2}{27}\left(\frac{1}{3}\right) = \frac{2}{81}$

15. $a_1 = 5,\ r = -\frac{1}{10}$

$a_1 = 5$

$a_2 = 5\left(-\frac{1}{10}\right) = -\frac{1}{2}$

$a_3 = \left(-\frac{1}{2}\right)\left(-\frac{1}{10}\right) = \frac{1}{20}$

$a_4 = \frac{1}{20}\left(-\frac{1}{10}\right) = -\frac{1}{200}$

$a_5 = \left(-\frac{1}{200}\right)\left(-\frac{1}{10}\right) = \frac{1}{2000}$

16. $a_1 = 6, r = -\frac{1}{4}$

$a_1 = 6$

$a_2 = 6\left(-\frac{1}{4}\right)^1 = -\frac{3}{2}$

$a_3 = 6\left(-\frac{1}{4}\right)^2 = \frac{3}{8}$

$a_4 = 6\left(-\frac{1}{4}\right)^3 = -\frac{3}{32}$

$a_5 = 6\left(-\frac{1}{4}\right)^4 = \frac{3}{128}$

17. $a_1 = 1,\ r = e$

$a_1 = 1$

$a_2 = 1(e) = e$

$a_3 = (e)(e) = e^2$

$a_4 = (e^2)(e) = e^3$

$a_5 = (e^3)(e) = e^4$

18. $a_1 = 4, r = \sqrt{3}$

$a_2 = 4\sqrt{3}$

$a_3 = 4\sqrt{3}\left(\sqrt{3}\right) = 12$

$a_4 = 12\left(\sqrt{3}\right) = 12\sqrt{3}$

$a_5 = 12\sqrt{3}\left(\sqrt{3}\right) = 36$

19. $a_1 = 64,\ a_{k+1} = \frac{1}{2}a_k$

$a_1 = 64$

$a_2 = \frac{1}{2}(64) = 32$

$a_3 = \frac{1}{2}(32) = 16$

$a_4 = \frac{1}{2}(16) = 8$

$a_5 = \frac{1}{2}(8) = 4$

$r = \frac{1}{2}, a_n = 64\left(\frac{1}{2}\right)^{n-1} = 128\left(\frac{1}{2}\right)^n$

20. $a_1 = 81, a_{k+1} = \frac{1}{3}a_k$

$a_1 = 81$

$a_2 = \frac{1}{3}(81) = 27$

$a_3 = \frac{1}{3}(27) = 9$

$a_4 = \frac{1}{3}(9) = 3$

$a_5 = \frac{1}{3}(3) = 1$

$r = \frac{1}{3}, a_n = 243\left(\frac{1}{3}\right)^n$

21. $a_1 = 9, a_{k+1} = 2a_k$

$a_2 = 2(9) = 18$

$a_3 = 2(18) = 36$

$a_4 = 2(36) = 72$

$a_5 = 2(72) = 144$

$r = 2$

$a_n = \left(\frac{9}{2}\right)2^n = 9(2^{n-1})$

22. $a_1 = 5, a_{k+1} = -3a_k$

$a_2 = -3(5) = -15$

$a_3 = -15(-3) = 45$

$a_4 = 45(-3) = -135$

$a_5 = -135(-3) = 405$

$r = -3, a_n = 5(-3)^{n-1}$

23. $a_1 = 6,\ a_{k+1} = -\frac{3}{2}a_k$

$$a_1 = 6$$
$$a_2 = -\frac{3}{2}(6) = -9$$
$$a_3 = -\frac{3}{2}(-9) = \frac{27}{2}$$
$$a_4 = -\frac{3}{2}\left(\frac{27}{2}\right) = -\frac{81}{4}$$
$$a_5 = -\frac{3}{2}\left(-\frac{81}{4}\right) = \frac{243}{8}$$
$$r = -\frac{3}{2},\ a_n = 6\left(-\frac{3}{2}\right)^{n-1} = -4\left(-\frac{3}{2}\right)^n$$

24. $a_1 = 30,\ a_{k+1} = -\frac{2}{3}a_k$

$$a_2 = \frac{-2}{3}a_1 = \frac{-2}{3}(30) = -20$$
$$a_3 = \frac{-2}{3}(-20) = \frac{40}{3}$$
$$a_4 = \frac{-2}{3}\left(\frac{40}{3}\right) = \frac{-80}{9}$$
$$a_5 = \frac{-2}{3}\left(\frac{-80}{9}\right) = \frac{160}{27}$$
$$r = \frac{-2}{3},\ a_n = 30\left(\frac{-2}{3}\right)^{n-1}$$

25. $a_1 = 4,\ a_4 = \frac{1}{2},\ n = 10$

$$a_1r^3 = a_4$$
$$4r^3 = \frac{1}{2}$$
$$r^3 = \frac{1}{8}$$
$$r = \frac{1}{2}$$
$$a_{10} = a_4r^6 = \frac{1}{2}\left(\frac{1}{2}\right)^6 = \frac{1}{2^7} = \frac{1}{128}$$

26. $a_1 = 5,\ a_3 = \frac{45}{4},\ n = 8$

$$a_1r^2 = a_3$$
$$5r^2 = \frac{45}{4}$$
$$r^2 = \frac{9}{4}$$
$$r = \pm\frac{3}{2}$$
$$a_8 = a_3r^5 = \frac{45}{4}\left(\pm\frac{3}{2}\right)^5 = \pm\frac{10,935}{128}$$

27. $a_1 = 6,\ r = -\frac{1}{3},\ n = 12$

$$a_n = a_1r^{n-1}$$
$$a_{12} = 6\left(-\frac{1}{3}\right)^{11} = \frac{-2}{3^{10}}$$

28. $a_1 = 8,\ r = \frac{-3}{4}$

$$a_n = a_1r^{n-1}$$
$$a_9 = 8\left(\frac{-3}{4}\right)^8 = \frac{6561}{8192}$$

29. $a_1 = 500,\ r = 1.02,\ n = 14$

$$a_n = a_1r^{n-1}$$
$$a_{14} = 500(1.02)^{13} \approx 646.8$$

30. $a_1 = 1000,\ r = 1.005,$
$n = 11$

$$a_n = a_1r^{n-1}$$
$$a_{11} = 1000(1.005)^{10} \approx 1051.14$$

31. $a_2 = a_1r = -18 \implies a_1 = \frac{-18}{r}$

$$a_5 = a_1r^4 = (a_1r)r^3 = -18r^3 = \frac{2}{3} \implies r = -\frac{1}{3}$$
$$a_1 = \frac{-18}{r} = \frac{-18}{-1/3} = 54$$
$$a_6 = a_1r^5 = 54\left(\frac{-1}{3}\right)^5 = \frac{-54}{243} = -\frac{2}{9}$$

32. $a_3 = \frac{16}{3},\ a_5 = \frac{64}{27},\ n = 7$

$a_3 r^2 = a_5$

$\frac{16}{3} r^2 = \frac{64}{27}$

$r^2 = \frac{4}{9}$

$r = \pm\frac{2}{3}$

$a_7 = a_5 r^2 = \frac{64}{27}\left(\pm\frac{2}{3}\right)^2 = \frac{256}{243}$

33. 7, 21, 63

$r = 3$

$a_n = 7(3)^{n-1}$

$a_9 = 7(3)^{9-1} = 45{,}927$

34. 3, 36, 432

$r = \frac{36}{3} = 12$

$a_n = 3(12)^{n-1}$

$a_7 = 3(12)^{7-1} = 8{,}957{,}952$

35. 5, 30, 180

$r = \frac{30}{5} = 6$

$a_n = 5(6)^{n-1}$

$a_{10} = 5(6)^{10-1} = 50{,}388{,}480$

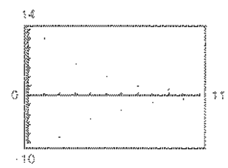
36. 4, 8, 16

$r = \frac{8}{4} = 2$

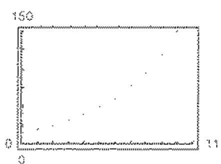
$a_n = 4(2)^{n-1}$

$a_{22} = 4(2)^{22-1} = 8{,}388{,}608$

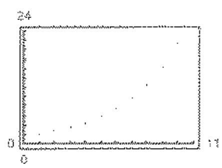
37. $a_n = 12(-0.75)^{n-1}$



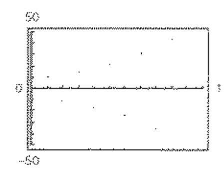
38.



39. $a_n = 2(1.3)^{n-1}$



40.



41. $8,\ -4,\ 2,\ -1,\ \frac{1}{2}$

$S_1 = 8$

$S_2 = 8 + (-4) = 4$

$S_3 = 8 + (-4) + 2 = 6$

$S_4 = 8 + (-4) + 2 + (-1) = 5$

42. $8, 12, 18, 27, \frac{81}{2}, \ldots$

$S_1 = 8$

$S_2 = 8 + 12 = 20$

$S_3 = 8 + 12 + 18 = 38$

$S_4 = 8 + 12 + 18 + 27 = 65$

43. $\sum_{n=1}^{\infty} 16\left(-\frac{1}{2}\right)^{n-1}$

n	1	2	3	4	5	6	7	8	9	10
S_n	16	24	28	30	31	31.5	31.75	31.875	31.9375	31.96875

44. $\sum_{n=1}^{\infty} 4(0.2)^{n-1}$

n	1	2	3	4	5	6	7	8	9	10
S_n	4	4.8	4.96	4.992	4.9984	4.99968	4.999936	4.9999872	≈ 5	≈ 5

45. $\sum_{n=1}^{9} 2^{n-1} \implies a_1 = 1,\ r = 2$

$$S_9 = \frac{1(1 - 2^9)}{1 - 2} = 511$$

46. $\sum_{n=1}^{9} (-2)^{n-1} \implies a_1 = 1, r = -2$

$$S_9 = \frac{1(1 - (-2)^9)}{1 - (-2)} = 171$$

47. $\sum_{i=1}^{7} 64\left(-\frac{1}{2}\right)^{i-1} \Rightarrow a_1 = 64,\ r = -\frac{1}{2}$

$S_7 = 64\left[\frac{1-(-1/2)^7}{1-(-1/2)}\right] = \frac{128}{3}\left[1-\left(-\frac{1}{2}\right)^7\right] = 43$

48. $\sum_{i=1}^{6} 32\left(\frac{1}{4}\right)^{i-1} \Rightarrow a_1 = 32, r = \frac{1}{4}$

$S_6 = 32\frac{(1-(1/4)^6)}{1-(1/4)} = \frac{1365}{32}$

49. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^n = \sum_{n=1}^{21} 3\left(\frac{3}{2}\right)^{n-1} \Rightarrow a_1 = 3,\ r = \frac{3}{2}$

$S_{21} = 3\left[\frac{1-(3/2)^{21}}{1-(3/2)}\right]$

$= -6\left[1-\left(\frac{3}{2}\right)^{21}\right] \approx 29{,}921.31$

50. $\sum_{n=0}^{15} 2\left(\frac{4}{3}\right)^n = \sum_{n=1}^{16} 2\left(\frac{4}{3}\right)^{n-1} \Rightarrow a_1 = 2, r = \frac{4}{3}$

$S_{16} = 2\left(\frac{1-(4/3)^{16}}{1-(4/3)}\right) \approx 592.65$

51. $\sum_{i=1}^{10} 8\left(-\frac{1}{4}\right)^{i-1} \Rightarrow a_1 = 8,\ r = -\frac{1}{4}$

$S_{10} = 8\left[\frac{1-(-1/4)^{10}}{1-(-1/4)}\right] = \frac{32}{5}\left[1-\left(-\frac{1}{4}\right)^{10}\right] \approx 6.4$

52. $\sum_{i=1}^{10} 5\left(-\frac{1}{3}\right)^{i-1} \Rightarrow a_1 = 5, r = -\frac{1}{3}$

$S_{10} = 5\left(\frac{1-(-1/3)^{10}}{1-(-1/3)}\right) \approx 3.75$

53. $\sum_{n=0}^{5} 300(1.06)^n = \sum_{n=1}^{6} 300(1.06)^{n-1} \Rightarrow a_1 = 300,\ r = 1.06$

$S_6 = 300\left[\frac{1-(1.06)^6}{1-1.06}\right] \approx 2092.60$

54. $\sum_{n=0}^{6} 500(1.04)^n = \sum_{n=1}^{7} 500(1.04)^{n-1} \Rightarrow a_1 = 500, r = 1.04$

$S_7 = 500\left(\frac{1-(1.04)^7}{1-1.04}\right) \approx 3949.15$

55. $5 + 15 + 45 + \cdots + 3645$

$r = 3$ and $3645 = 5(3)^{n-1} \Rightarrow n = 7$

Thus, the sum can be written as $\sum_{n=1}^{7} 5(3)^{n-1}$.

56. $7 + 14 + 28 + \cdots + 896$

$r = 2$ and $896 = 7(2)^{n-1} \Rightarrow n = 8$

$\sum_{n=1}^{8} 7(2)^{n-1}$

57. $2 - \frac{1}{2} + \frac{1}{8} - \cdots + \frac{1}{2048}$

$r = -\frac{1}{4}$ and $\frac{1}{2048} = 2\left(-\frac{1}{4}\right)^{n-1} \Rightarrow n = 7$

$\sum_{n=1}^{7} 2\left(-\frac{1}{4}\right)^{n-1}$

58. $15 - 3 + \frac{3}{5} - \cdots - \frac{3}{625}$

$r = -0.2$ and $-\frac{3}{625} = 15(-0.2)^{n-1} \Rightarrow n = 6$

$\sum_{n=1}^{6} 15(-0.2)^{n-1}$

59. $a_1 = 10, r = \frac{4}{5}$

$\sum_{n=0}^{\infty} 10\left(\frac{4}{5}\right)^n = \frac{a_1}{1-r} = \frac{10}{1-\frac{4}{5}} = 50$

60. $a_1 = 6, r = \frac{2}{3}$

$\sum_{n=0}^{\infty} 6\left(\frac{2}{3}\right)^n = \frac{a_1}{1-r} = \frac{6}{1-\frac{2}{3}} = 18$

61. $a_1 = 5, r = -\frac{1}{2}$

$$\sum_{n=0}^{\infty} 5\left(-\frac{1}{2}\right)^n = \frac{a_1}{1-r} = \frac{5}{1-\left(-\frac{1}{2}\right)} = \frac{5}{\left(\frac{3}{2}\right)} = \frac{10}{3}$$

62. $a_1 = 9, r = -\frac{2}{3}$

$$\sum_{n=0}^{\infty} 9\left(-\frac{2}{3}\right)^n = \frac{a_1}{1-r} = \frac{9}{1-\left(-\frac{2}{3}\right)} = \frac{9}{\left(\frac{5}{3}\right)} = \frac{27}{5}$$

63. $\sum_{n=1}^{\infty} 2\left(\frac{7}{3}\right)^{n-1}$ does not have a finite sum $\left(\frac{7}{3} > 1\right)$.

64. $\sum_{n=1}^{\infty} 8\left(\frac{5}{3}\right)^{n-1}$ does not have a finite sum $\left(\frac{5}{3} > 1\right)$.

65. $a_1 = 10, r = 0.11$

$$\sum_{n=0}^{\infty} 10(0.11)^n = \frac{a_1}{1-r} = \frac{10}{1-0.11} = \frac{10}{0.89}$$
$$= \frac{1000}{89} \approx 11.236$$

66. $a_1 = 5, r = 0.45$

$$\sum_{n=0}^{\infty} 5(0.45)^n = \frac{a_1}{1-r} = \frac{5}{1-0.45} = \frac{5}{0.55} = \frac{100}{11}$$
$$\approx 9.091$$

67. $a_1 = -3, r = -0.9$

$$\sum_{n=0}^{\infty} -3(-0.9)^n = \frac{a_1}{1-r} = \frac{-3}{1-(-0.9)}$$
$$= \frac{-3}{1.9} = \frac{-30}{19} \approx -1.579$$

68. $a_1 = -10, r = -0.2$

$$\sum_{n=0}^{\infty} -10(-0.2)^n = \frac{a_1}{1-r} = \frac{-10}{1-(-0.2)}$$
$$= \frac{-10}{1.2} = \frac{-25}{3} \approx -8.333$$

69. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots = \sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n$

$$= \frac{8}{1-3/4} = 32$$

70. $9 + 6 + 4 + \frac{8}{3} + \cdots = \sum_{n=0}^{\infty} 9\left(\frac{2}{3}\right)^n$

$$= \frac{9}{1-2/3} = \frac{9}{1/3} = 27$$

71. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots = \sum_{n=0}^{\infty} 3\left(-\frac{1}{3}\right)^n = \frac{a_1}{1-r} = \frac{3}{1-(-1/3)} = 3\left(\frac{3}{4}\right) = \frac{9}{4}$

72. $-6 + 5 - \frac{25}{6} + \frac{125}{36} - \cdots = \sum_{n=0}^{\infty} -6\left(-\frac{5}{6}\right)^n = \frac{-6}{1-(-5/6)} = \frac{-6}{11/6} = \frac{-36}{11} \approx -3.2727$

73. $0.\overline{36} = \sum_{n=0}^{\infty} 0.36(0.01)^n$

$$= \frac{0.36}{1-0.01} = \frac{0.36}{0.99} = \frac{36}{99} = \frac{4}{11}$$

74. $0.\overline{297} = \sum_{n=0}^{\infty} 0.297(0.001)^n$

$$= \frac{0.297}{1-0.001} = \frac{0.297}{0.999} = \frac{297}{999} = \frac{11}{37}$$

75. $1.2\overline{5} = 1.2 + \sum_{n=0}^{\infty} 0.05(0.1)^n$

$$= \frac{6}{5} + \frac{0.05}{1-0.1}$$
$$= \frac{6}{5} + \frac{0.05}{0.9}$$
$$= \frac{6}{5} + \frac{5}{90} = \frac{113}{90}$$

76. $1.3\overline{8} = 1.3 + \sum_{n=0}^{\infty} 0.08(0.1)^n$

$$= 1.3 + \frac{0.08}{1-0.1}$$
$$= 1.3 + \frac{0.08}{0.9}$$
$$= 1\frac{3}{10} + \frac{4}{45} = 1\frac{7}{18} = \frac{25}{18}$$

77. $A = P\left(1 + \frac{r}{n}\right)^{nt} = 1000\left(1 + \frac{0.03}{n}\right)^{n(10)}$

(a) $n = 1$: $A = 1000(1 + 0.03)^{10} \approx 1343.92$

(b) $n = 2$: $A = 1000\left(1 + \frac{0.03}{2}\right)^{2(10)} \approx 1346.86$

(c) $n = 4$: $A = 1000\left(1 + \frac{0.03}{4}\right)^{4(10)} \approx 1348.35$

(d) $n = 12$: $A = 1000\left(1 + \frac{0.03}{12}\right)^{12(10)} \approx 1349.35$

(e) $n = 365$: $A = 1000\left(1 + \frac{0.03}{365}\right)^{365(10)} \approx 1349.84$

78. $A = P\left(1 + \frac{r}{n}\right)^{nt} = 2500\left(1 + \frac{0.04}{n}\right)^{20n}$

(a) $n = 1, A = 2500\left(1 + \frac{0.04}{1}\right)^{20(1)} = 5477.81$

(b) $n = 2, A = 5520.10$

(c) $n = 4, A = 5541.79$

(d) $n = 12, A = 5556.46$

(e) $n = 365, A = 5563.61$

79. $A = \sum_{n=1}^{60} 100\left(1 + \frac{0.03}{12}\right)^n$

$= 100\left(1 + \frac{0.03}{12}\right) \cdot \frac{[1 - (1 + 0.03/12)^{60}]}{[1 - (1 + 0.03/12)]}$

$= 100(1.0025) \cdot \left[\frac{1 - 1.0025^{60}}{1 - 1.0025}\right]$

$\approx \$6480.83$

80. $A = \sum_{n=1}^{60} 50\left(1 + \frac{0.02}{12}\right)^n$

$= 50\left(1 + \frac{0.02}{12}\right) \cdot \frac{[1 - (1 + 0.02/12)^{60}]}{[1 - (1 + 0.02/12)]}$

$\approx \$3157.62$

81. Let $N = 12t$ be the total number of deposits.

$$A = P\left(1 + \frac{r}{12}\right) + P\left(1 + \frac{r}{12}\right)^2 + \cdots + P\left(1 + \frac{r}{12}\right)^N$$

$$= \left(1 + \frac{r}{12}\right)\left[P + P\left(1 + \frac{r}{12}\right) + \cdots + P\left(1 + \frac{r}{12}\right)^{N-1}\right]$$

$$= P\left(1 + \frac{r}{12}\right)\sum_{n=1}^{N}\left(1 + \frac{r}{12}\right)^{n-1}$$

$$= P\left(1 + \frac{r}{12}\right)\frac{1 - \left(1 + \frac{r}{12}\right)^N}{1 - \left(1 + \frac{r}{12}\right)}$$

$$= P\left(1 + \frac{r}{12}\right)\left(-\frac{12}{r}\right)\left[1 - \left(1 + \frac{r}{12}\right)^N\right]$$

$$= P\left(\frac{12}{r} + 1\right)\left[-1 + \left(1 + \frac{r}{12}\right)^N\right]$$

$$= P\left[\left(1 + \frac{r}{12}\right)^N - 1\right]\left(1 + \frac{12}{r}\right)$$

$$= P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right)$$

82. Let $N = 12t$ be the total number of deposits.

$$A = Pe^{r/12} + Pe^{2r/12} + \cdots + Pe^{Nr/12}$$

$$= \sum_{n=1}^{N} Pe^{r/12 \cdot n}$$

$$= Pe^{r/12}\frac{(1 - (e^{r/12})^N)}{(1 - e^{r/12})}$$

$$= Pe^{r/12}\frac{(1 - (e^{r/12})^{12t})}{1 - e^{r/12}}$$

$$= \frac{Pe^{r/12}(e^{rt} - 1)}{(e^{r/12} - 1)}$$

83. $P = \$50,\ r = 7\%,\ t = 20$ years

(a) Compounded monthly: $A = 50\left[\left(1 + \frac{0.07}{12}\right)^{12(20)} - 1\right]\left(1 + \frac{12}{0.07}\right) \approx \$26{,}198.27$

(b) Compounded continuously: $A = \frac{50e^{0.07/12}(e^{0.07(20)} - 1)}{e^{0.07/12} - 1} \approx \$26{,}263.88$

84. $P = 75, r = 0.04, t = 25$

(a) $A = 75\left[\left(1 + \frac{0.04}{12}\right)^{12(25)} - 1\right]\left(1 + \frac{12}{0.04}\right) \approx \$38{,}688.25$

(b) $A = \frac{75e^{0.04/12}(e^{0.04(25)} - 1)}{e^{0.04/12} - 1} \approx \$38{,}725.81$

85. $P = 100, r = 5\% = 0.05, t = 40$

(a) Compounded monthly: $A = 100\left[\left(1 + \frac{0.05}{12}\right)^{12(40)} - 1\right]\left(1 + \frac{12}{0.05}\right) \approx \$153{,}237.86$

(b) Compounded continuously: $A = \frac{100e^{0.05/12}(e^{0.05(40)} - 1)}{e^{0.05/12} - 1} \approx \$153{,}657.02$

86. $P = \$20, r = 6\%, t = 50$ years

(a) Compounded monthly: $A = 20\left[\left(1 + \frac{0.06}{12}\right)^{12(50)} - 1\right]\left(1 + \frac{12}{0.06}\right) \approx \$76{,}122.54$

(b) Compounded continuously: $A = \frac{20e^{0.06/12}(e^{0.06(50)} - 1)}{e^{0.06/12} - 1} \approx \$76{,}533.16$

87. First shaded area: $\frac{16^2}{4}$ Second shaded area: $\frac{16^2}{4} + \frac{1}{2}\cdot\frac{16^2}{4}$ Third shaded area: $\frac{16^2}{4} + \frac{1}{2}\frac{16^2}{4} + \frac{1}{4}\frac{16^2}{4}$, etc.

Total area of shaded region: $\frac{16^2}{4}\sum_{n=0}^{5}\left(\frac{1}{2}\right)^n = 64\left[\frac{1 - (1/2)^6}{1 - 1/2}\right] = 128\left(1 - \left(\frac{1}{2}\right)^6\right) = 126$ square units

88. $27^2\left(\frac{1}{9}\right) + 27^2\left(\frac{1}{9}\right)\left(\frac{8}{9}\right) + 27^2\left(\frac{1}{9}\right)\left(\frac{8}{9}\right)^2 + 27^2\left(\frac{1}{9}\right)\left(\frac{8}{9}\right)^3 = \sum_{n=0}^{3} 27^2\left(\frac{1}{9}\right)\left(\frac{8}{9}\right)^n = \frac{2465}{9} \approx 273.89$ square inches

89. (a) $a_0 = 70$ degrees

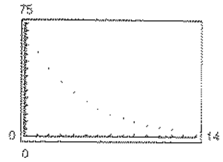
$a_1 = 0.8(70) = 56$ degrees

$\vdots$

$a_n = (0.8)^n(70)$

(b) $a_6 = (0.8)^6(70) \approx 18.35$ degrees

$a_{12} = (0.8)^{12}(70) \approx 4.81$ degrees

(c) 

$a_3 \approx 35.8$

$a_4 \approx 28.7$

Thus, the water freezes between 3 and 4 hours, about 3.5 hours.

90. (a) Surface area of a sphere is $4\pi r^2$. The surface area of the sphere flake is

$$S = 4\pi(1)^2 + 9\left(4\pi\left(\frac{1}{3}\right)^2\right) + 9^2\left(4\pi\left(\frac{1}{9}\right)^2\right) + \cdots = 4\pi + 4\pi + 4\pi + \cdots = \sum_{n=1}^{\infty} 4\pi.$$

(b) Volume of a sphere is $\frac{4}{3}\pi r^2$. The volume of the sphere flake is

$$V = \frac{4}{3}\pi(1)^3 + 9\left(\frac{4}{3}\pi\left(\frac{1}{3}\right)^3\right) + 9^2\left(\frac{4}{3}\pi\left(\frac{1}{9}\right)^3\right) + \cdots = \frac{4}{3}\pi + \frac{4}{3}\pi\left(\frac{1}{3}\right) + \frac{4}{3}\pi\left(\frac{1}{3}\right)^2 + \cdots = \sum_{n=0}^{\infty} \frac{4}{3}\pi\left(\frac{1}{3}\right)^n.$$

(c) The surface area is infinite and the volume is finite.

$$V = \frac{4/3\pi}{1 - 1/3} = 2\pi$$

91. $400 + 0.75(400) + (0.75)^2(400) + \cdots = \sum_{n=0}^{\infty} 400(0.75)^n$

$$= \frac{400}{1 - 0.75} = \$1600$$

92. $500 + 0.70(500) + (0.70)^2(500) + \cdots = \sum_{n=0}^{\infty} 500(0.70)^n$

$$= \frac{500}{1 - 0.70} \approx \$1666.67$$

93. $250 + 0.80(250) + (0.80)^2(250) + \cdots = \sum_{n=0}^{\infty} 250(0.80)^2$

$$= \frac{250}{1 - 0.80} = \$1250$$

94. $350 + 0.75(350) + (0.75)^2(350) + \cdots = \sum_{n=0}^{\infty} 350(0.75)^n$

$$= \frac{350}{1 - 0.75} = \$1400$$

95. $600 + 0.725(600) + (0.725)^2(600) + \cdots = \sum_{n=0}^{\infty} 600(0.725)^n$

$$= \frac{600}{1 - 0.725} \approx \$2181.82$$

96. $450 + 0.775(450) + (0.775)^2(450) + \cdots = \sum_{n=0}^{\infty} 450(0.775)^n$

$$= \frac{450}{1 - 0.775} = \$2000$$

97. (a) Option 1: $30{,}000 + 1.025(30{,}000) + \cdots + (1.025)^4(30{,}000) = \sum_{n=0}^{4} 30{,}000(1.025)^n$

$\approx \$157{,}689.86$

Option 2: $32{,}500 + 1.02(32{,}500) + \cdots + (1.02)^4(32{,}500) = \sum_{n=0}^{4} 32{,}500(1.02)^n$

$\approx \$169{,}131.31$

Option 2 has the larger cumulative amount.

(b) Option 1: $(1.025)^4(30{,}000) \approx \$33{,}114.39$

Option 2: $(1.02)^4(32{,}500) \approx \$35{,}179.05$

Option 2 has the larger amount.

98. (a) $8000 + 0.9(8000) + \ldots + (0.9)^{n-1}(8000) = \sum_{i=0}^{n-1} 8000(0.9)^i$

(b) $\sum_{i=0}^{9} 8000(0.9)^i = 52{,}106$ units (10 years)

$\sum_{i=0}^{19} 8000(0.9)^i = 70{,}274$ units (20 years)

$\sum_{i=0}^{49} 8000(0.9)^i = 79{,}588$ units (50 years)

(c) $\sum_{i=0}^{\infty} 8000(0.9)^i = \dfrac{8000}{1 - 0.9} = 80{,}000$

If this trend continues indefinitely, the number of units will be 80,000.

99. (a) Downward: $850 + 0.75(850) + (0.75)^2(850) + \cdots + (0.75)^9(850) = \sum_{n=0}^{9} 850(0.75)^n$

≈ 3208.53 feet

Upward: $0.75(850) + (0.75)^2(850) + \cdots + (0.75)^{10}(850) = \sum_{n=0}^{9} (0.75)(850)(0.75)^n$

$= \sum_{n=0}^{9} 637.5(0.75)^n \approx 2406.4$ feet

Total distance: $3208.53 + 2406.4 = 5614.93$ feet

(b) $\sum_{n=0}^{\infty} 850(0.75)^n + \sum_{n=0}^{\infty} 637.5(0.75)^n = \dfrac{850}{1 - 0.75} + \dfrac{637.5}{1 - 0.75} = 5950$ feet

100. (a) Total distance $= \sum_{n=0}^{\infty} 32(0.81)^n - 16 = \dfrac{32}{1 - 0.81} - 16 \approx 152.42$ feet

(b) Total time $= 1 + 2\sum_{n=1}^{\infty} (0.9)^n = 1 + 2\left[\dfrac{1}{1 - 0.9} - 1\right] = 19$ seconds

101. False. See definition page 535.

102. False. You multiply the first term by the common ratio raised to the $(n - 1)$ power.

103. $a_1 = 3,\ r = \dfrac{x}{2}$

$$a_2 = 3\left(\frac{x}{2}\right) = \frac{3x}{2}$$

$$a_3 = \frac{3x}{2}\left(\frac{x}{2}\right) = \frac{3x^2}{4}$$

$$a_4 = \frac{3x^2}{4}\left(\frac{x}{2}\right) = \frac{3x^3}{8}$$

$$a_5 = \frac{3x^3}{8}\left(\frac{x}{2}\right) = \frac{3x^4}{16}$$

104. $a_1 = \dfrac{1}{2}$

$$a_2 = \frac{1}{2}(7x) = \frac{7x}{2}$$

$$a_3 = \frac{7x}{2}(7x) = \frac{7^2x^2}{2}$$

$$a_4 = \frac{7^3x^3}{2}$$

$$a_5 = \frac{7^4x^4}{2}$$

105. $a_1 = 100,\ r = e^x,\ n = 9$

$$a_n = a_1r^{n-1}$$

$$a_9 = 100(e^x)^8 = 100e^{8x}$$

106. $a_1 = 4,\ r = \dfrac{4x}{3},\ n = 6$

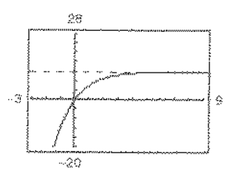
$$a_n = a_1r^{n-1}$$

$$a_6 = 4\left(\frac{4x}{3}\right)^5 = \frac{4096}{243}x^5$$

107. (a) $f(x) = 6\left[\dfrac{1 - 0.5^x}{1 - 0.5}\right]$

$$\sum_{n=0}^{\infty} 6\left(\frac{1}{2}\right)^n = \frac{6}{1 - 1/2} = 12$$

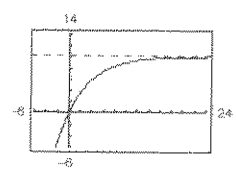
The horizontal asymptote of $f(x)$ is $y = 12$. This corresponds to the sum of the series.



(b) $f(x) = 2\left[\dfrac{1 - 0.8^x}{1 - 0.8}\right]$

$$\sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n = \frac{2}{1 - 4/5} = 10$$

The horizontal asymptote of $f(x)$ is $y = 10$. This corresponds to the sum of the series.



108. Given a real number r between -1 and 1, $|a_n| = |a_{n-1}(r)| < |a_{n-1}|$ which shows that the terms decrease.

109. To use the first two terms of a geometric series to find the nth term, first divide the second term by the first term to obtain the constant ratio. The nth term is the first term multiplied by the common ratio raised to the $(n - 1)$ power.

$$r = \frac{a_2}{a_1},\ a_n = a_1r^{n-1}$$

110. a_1

$$a_2 = a_1r$$

$$a_3 = a_2r = a_1r^2$$

$$a_4 = a_1r^3$$

$$a_n = a_1r^{n-1}$$

111. $\text{Time} = \dfrac{\text{Distance}}{\text{Speed}} = \dfrac{200}{50} + \dfrac{200}{42} = 200\left[\dfrac{92}{2100}\right]$ hours

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{400}{200[92/2100]} = \frac{2(2100)}{92} \approx 45.65 \text{ mph}$$

112. Your friend mows at the rate of $\frac{1}{4}$ lawns/hour, and your rate is $\frac{1}{6}$ lawns/hour. Together, the time would be

$$\frac{1}{(1/4)+(1/6)} = \frac{1}{10/24} = \frac{24}{10} = 2.4 \text{ hours.}$$

113. $\det\begin{bmatrix} -1 & 3 & 4 \\ -2 & 8 & 0 \\ 2 & 5 & -1 \end{bmatrix} = 4(-10-16) - 1(-8+6)$

$= -104 + 2 = -102$

114. $\det\begin{bmatrix} -1 & 0 & 4 \\ -4 & 3 & 5 \\ 0 & 2 & -3 \end{bmatrix} = -1(-9-10) + 4(-8-0)$

$= 19 - 32 = -13$

115. Answers will vary.

Section 8.4 Mathematical Induction

- You should be sure that you understand the principle of mathematical induction. If P_n is a statement involving the positive integer n, where P_1 is true and the truth of P_k implies the truth of P_{k+1}, then P_n is true for all positive integers n.
- You should be able to verify (by induction) the formulas for the sums of powers of integers and be able to use these formulas.
- You should be able to work with finite differences.

Vocabulary Check

1. mathematical induction
2. first
3. arithmetic
4. second

1. $P_k = \dfrac{5}{k(k+1)}$

$P_{k+1} = \dfrac{5}{(k+1)[(k+1)+1]} = \dfrac{5}{(k+1)(k+2)}$

2. $P_k = \dfrac{4}{(k+2)(k+3)}$

$P_{k+1} = \dfrac{4}{[(k+1)+2][(k+1)+3]}$

$= \dfrac{4}{(k+3)(k+4)}$

3. $P_k = \dfrac{2^k}{(k+1)!}$

$P_{k+1} = \dfrac{2^{k+1}}{((k+1)+1)!} = \dfrac{2^{k+1}}{(k+2)!}$

4. $P_k = \dfrac{2^{k-1}}{k!}$

$P_{k+1} = \dfrac{2^{(k+1)-1}}{(k+1)!} = \dfrac{2^k}{(k+1)!}$

5. $P_k = 1 + 6 + 11 + \cdots + [5(k-1)-4] + [5k-4]$

$P_{k+1} = 1 + 6 + 11 + \cdots + [5k-4] + [5(k+1)-4]$

$= 1 + 6 + 11 + \cdots + [5k-4] + [5k+1]$

6. $P_k = 7 + 13 + 19 + \cdots + [6(k-1)+1] + (6k+1)$

$$P_{k+1} = 7 + 13 + 19 + \cdots + (6k+1) + (6(k+1)+1)$$
$$= 7 + 13 + 19 + \cdots + (6k+1) + (6k+7)$$

7. 1. When $n = 1$, $S_1 = 2 = 1(1+1)$.

2. Assume that $S_k = 2 + 4 + 6 + 8 + \cdots + 2k = k(k+1)$.

Then,

$$S_{k+1} = 2 + 4 + 6 + 8 + \cdots + 2k + 2(k+1)$$
$$= S_k + 2(k+1) = k(k+1) + 2(k+1) = (k+1)(k+2).$$

Therefore, by mathematical induction, the formula is valid for all positive integer values of n.

8. 1. When $n = 1$, $S_1 = 3 = 1(4(1) - 1) = 3$

2. Assume that $S_k = 3 + 11 + \cdots + (8k-5) = k(4k-1)$.

Then,

$$S_{k+1} = 3 + 11 + \cdots + (8k-5) + (8(k+1)-5)$$
$$= 3 + 11 + \cdots + (8k-5) + (8k+3)$$
$$= S_k + (8k+3)$$
$$= k(4k-1) + (8k+3)$$
$$= 4k^2 + 7k + 3$$
$$= (k+1)(4k+3)$$
$$= (k+1)(4(k+1)-1).$$

Therefore, by mathematical induction, the formula is valid for all $n \geq 1$.

9. 1. When $n = 1$, $S_1 = 3 = \frac{1}{2}(5(1)+1)$

2. Assume that $S_k = 3 + 8 + 13 + \cdots + (5k-2) = \frac{k}{2}(5k+1)$.

Then,

$$S_{k+1} = 3 + 8 + 13 + \cdots + (5k-2) + [5(k+1)-2]$$
$$= S_k + [5k+3] = \frac{k}{2}(5k+1) + 5k + 3$$
$$= \frac{1}{2}[5k^2 + 11k + 6] = \frac{1}{2}(k+1)(5k+6)$$
$$= \frac{1}{2}(k+1)(5(k+1)+1).$$

Therefore, by mathematical induction, the formula is valid for all positive integer values of n.

10. 1. When $n = 1$,

$$S_1 = 1 = \frac{1}{2}(3 \cdot 1 - 1).$$

2. Assume that $S_k = 1 + 4 + 7 + 10 + \cdots + (3k - 2) = \frac{k}{2}(3k - 1)$.

Then,

$$\begin{aligned} S_{k+1} &= 1 + 4 + 7 + 10 + \cdots + (3k - 2) + (3(k + 1) - 2) \\ &= S_k + (3(k + 1) - 2) \\ &= \frac{k}{2}(3k - 1) + (3k + 1) \\ &= \frac{3k^2 - k + 6k + 2}{2} \\ &= \frac{3k^2 + 5k + 2}{2} \\ &= \frac{(k + 1)(3k + 2)}{2} \\ &= \frac{k + 1}{2}[3(k + 1) - 1]. \end{aligned}$$

Therefore, the formula is valid for all positive integer values of n.

11. 1. When $n = 1$, $S_1 = 1 = 2^1 - 1$.

2. Assume that

$S_k = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} = 2^k - 1$.

Then,

$$\begin{aligned} S_{k+1} &= 1 + 2 + 2^2 + 2^3 + \cdots + 2^{k-1} + 2^k \\ &= S_k + 2^k = 2^k - 1 + 2^k = 2(2^k) - 1 = 2^{k+1} - 1. \end{aligned}$$

Therefore, by mathematical induction, the formula is valid for all positive integer values of n.

12. 1. When $n = 1$, $S_1 = 2 = 3^1 - 1$.

2. Assume that $S_k = 2(1 + 3 + 3^2 + 3^3 + \cdots + 3^{k-1}) = 3^k - 1$.

Then,

$$\begin{aligned} S_{k+1} &= 2(1 + 3 + 3^2 + 3^3 + \cdots + 3^{k-1}) + 2 \cdot 3^{k+1-1} \\ &= S_k + 2 \cdot 3^k \\ &= 3^k - 1 + 2 \cdot 3^k \\ &= 3 \cdot 3^k - 1 \\ &= 3^{k+1} - 1. \end{aligned}$$

Therefore, the formula is valid for all positive integer values of n.

13. 1. When $n = 1$, $S_1 = 1 = \dfrac{1(1+1)}{2}$.

2. Assume that

$$S_k = 1 + 2 + 3 + 4 + \cdots + k = \frac{k(k+1)}{2}.$$

Then,

$$\begin{aligned} S_{k+1} &= 1 + 2 + 3 + 4 + \cdots + k + (k+1) \\ &= S_k + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Therefore, the formula is valid for all positive integer values of n.

14. 1. When $n = 1$, $S_1 = 1^3 = 1 = \dfrac{1(1+1)^2}{4}$.

2. Assume that $S_k = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$.

Then,

$$\begin{aligned} S_{k+1} &= 1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 \\ &= S_k + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} = \frac{(k+1)^2(k^2 + 4k + 4)}{4} = \frac{(k+1)^2(k+2)^2}{4}. \end{aligned}$$

Therefore, the formula is valid for all positive integer values of n.

15. 1. When $n = 1$,

$$S_1 = 1^4 = \frac{1(1+1)(2\cdot 1 + 1)(3\cdot 1^2 + 3\cdot 1 - 1)}{30}.$$

2. Assume that $S_k = \displaystyle\sum_{i=1}^{k} i^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$.

Then, $S_{k+1} = S_k + (k+1)^4$

$$\begin{aligned} &= \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1) + 30(k+1)^4}{30} \\ &= \frac{(k+1)[k(2k+1)(3k^2+3k-1) + 30(k+1)^3]}{30} = \frac{(k+1)(6k^4 + 39k^3 + 91k^2 + 89k + 30)}{30} \\ &= \frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30} = \frac{(k+1)(k+2)(2(k+1)+1)(3(k+1)^2 + 3(k+1) - 1)}{30}. \end{aligned}$$

Therefore, the formula is valid for all positive integer values of n.

16. 1. When $n = 1$, $S_1 = \dfrac{(1)^2(1+1)^2(2(1)^2+2(1)-1)}{12} = 1.$

2. Assume that $S_k = \displaystyle\sum_{i=1}^{k} i^5 = \frac{k^2(k+1)^2(2k^2+2k-1)}{12}.$

Then,

$$\begin{aligned}
S_{k+1} &= \sum_{i=1}^{k+1} i^5 = \sum_{i=1}^{k} i^5 + (k+1)^5 \\
&= \frac{k^2(k+1)^2(2k^2+2k-1)}{12} + \frac{12(k+1)^5}{12} \\
&= \frac{(k+1)^2[k^2(2k^2+2k-1)+12(k+1)^3]}{12} \\
&= \frac{(k+1)^2[2k^4+2k^3-k^2+12(k^3+3k^2+3k+1)]}{12} \\
&= \frac{(k+1)^2[2k^4+14k^3+35k^2+36k+12]}{12} \\
&= \frac{(k+1)^2(k^2+4k+4)(2k^2+6k+3)}{12} \\
&= \frac{(k+1)^2(k+2)^2[2(k+1)^2+2(k+1)-1]}{12}.
\end{aligned}$$

Therefore, the formula is valid for all positive integer values of n.

17. 1. When $n = 1$, $S_1 = 2 = \dfrac{1(2)(3)}{3}.$

2. Assume that $S_k = 1(2) + 2(3) + 3(4) + \cdots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}.$

Then,

$$\begin{aligned}
S_{k+1} &= 1(2) + 2(3) + 3(4) + \cdots + k(k+1) + (k+1)(k+2) \\
&= S_k + (k+1)(k+2) \\
&= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\
&= \frac{(k+1)(k+2)(k+3)}{3}.
\end{aligned}$$

Therefore, the formula is valid for all positive integer values of n.

18. 1. When $n = 1$, $S_1 = \dfrac{1}{(1)(3)} = \dfrac{1}{2 + 1}$.

2. Assume that $S_k = \displaystyle\sum_{i=1}^{k} \frac{1}{(2i - 1)(2i + 1)} = \frac{k}{2k + 1}$.

Then,

$$\begin{aligned}
S_{k+1} &= S_k + \frac{1}{(2(k + 1) - 1)(2(k + 1) + 1)} \\
&= \frac{k}{2k + 1} + \frac{1}{(2k + 1)(2k + 3)} \\
&= \frac{k(2k + 3) + 1}{(2k + 1)(2k + 3)} \\
&= \frac{2k^2 + 3k + 1}{(2k + 1)(2k + 3)} \\
&= \frac{(2k + 1)(k + 1)}{(2k + 1)(2k + 3)} \\
&= \frac{k + 1}{2(k + 1) + 1}.
\end{aligned}$$

Therefore, the formula is valid for all positive integer values of n.

19. 1. When $n = 1$, $S_1 = \dfrac{1}{1(1 + 1)} = \dfrac{1}{2}$.

2. Assume $S_k = \displaystyle\sum_{i=1}^{k} \frac{1}{i(i + 1)} = \frac{k}{k + 1}$.

Thus,

$$\begin{aligned}
S_{k+1} &= \sum_{i=1}^{k+1} \frac{1}{i(i + 1)} \\
&= \frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{k(k + 1)} + \frac{1}{(k + 1)(k + 2)} \\
&= \frac{k}{k + 1} + \frac{1}{(k + 1)(k + 2)} \\
&= \frac{k(k + 2) + 1}{(k + 2)(k + 2)} \\
&= \frac{(k + 1)^2}{(k + 1)(k + 2)} \\
&= \frac{k + 1}{k + 2} \\
&= \frac{k + 1}{(k + 1) + 1}.
\end{aligned}$$

Therefore, the formula is valid for all positive integer values of n.

20. 1. When $n = 1$, $\frac{1}{1(2)(3)} = \frac{1}{6} = \frac{1(4)}{4(2)(3)}$.

2. Assume $\sum_{i=1}^{k} \frac{1}{i(i+1)(i+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$.

Thus,

$$
\begin{aligned}
S_{k+1} &= \sum_{i=1}^{k+1} \frac{1}{i(i+1)(i+2)} \\
&= \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \\
&= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \\
&= \frac{k(k+3)(k+3) + 4}{4(k+1)(k+2)(k+3)} \\
&= \frac{k^3 + 6k^2 + 9k + 4}{4(k+1)(k+2)(k+3)} \\
&= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)} \\
&= \frac{(k+1)(k+4)}{4(k+2)(k+3)}.
\end{aligned}
$$

Therefore, the formula is valid for all positive integer values of n.

21. $\sum_{n=1}^{50} n^3 = \frac{50^2(50+1)^2}{4} = 1{,}625{,}625$

22.
$$
\begin{aligned}
\sum_{n=1}^{10} n^4 &= \frac{10(10+1)(2 \cdot 10 + 1)(3 \cdot 10^2 + 3 \cdot 10 - 1)}{30} \\
&= \frac{10(11)(21)(329)}{30} = 25{,}333
\end{aligned}
$$

23.
$$
\begin{aligned}
\sum_{n=1}^{12} (n^2 - n) &= \sum_{n=1}^{12} n^2 - \sum_{n=1}^{12} n \\
&= \frac{12(12+1)(2 \cdot 12 + 1)}{6} - \frac{12(12+1)}{2} \\
&= 650 - 78 = 572
\end{aligned}
$$

24.
$$
\begin{aligned}
\sum_{n=1}^{40} (n^3 - n) &= \sum_{n=1}^{40} n^3 - \sum_{n=1}^{40} n \\
&= \frac{40^2(40+1)^2}{4} - \frac{40(40+1)}{2} \\
&= 672{,}400 - 820 = 671{,}580
\end{aligned}
$$

25. 1. When $n = 4$, $4! = 24$ and $2^4 = 16$, thus $4! > 2^4$.

2. Assume $k! > 2^k$, $k > 4$. Then, $(k+1)! = k!(k+1) > 2^k(2)$ since $k + 1 > 2$. Thus, $(k+1)! > 2^{k+1}$.

Therefore, by mathematical induction, the formula is valid for all integers n such that $n \geq 4$.

26. 1. When $n = 7$, $\left(\frac{4}{3}\right)^7 \approx 7.4915 > 7$.

2. Assume that $\left(\frac{4}{3}\right)^k > k$, $k > 7$.

Then, $\left(\frac{4}{3}\right)^{k+1} = \left(\frac{4}{3}\right)^k\left(\frac{4}{3}\right) > k\left(\frac{4}{3}\right) = k + \frac{k}{3} > k + 1$ for $k > 7$. Thus, $\left(\frac{4}{3}\right)^{k+1} > k + 1$.

Therefore, $\left(\frac{4}{3}\right)^n > n$.

27. 1. When $n = 2$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \approx 1.707$ and $\sqrt{2} \approx 1.414$, thus $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$.

2. Assume $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k}$, $k > 2$.

Then, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$.

Now we need to show that $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$, $k > 2$.

This is true because $\sqrt{k(k+1)} > k$

$$\sqrt{k(k+1)} + 1 > k + 1$$

$$\frac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}} > \frac{k+1}{\sqrt{k+1}}$$

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}.$$

Therefore, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$.

Therefore, by mathematical induction, the formula is valid for all integers n such that $n \geq 2$.

28. 1. When $n = 1$, $\left(\frac{x}{y}\right)^2 < \left(\frac{x}{y}\right)$ and $(0 < x < y)$.

2. Assume that $\left(\frac{x}{y}\right)^{k+1} < \left(\frac{x}{y}\right)^k$

$$\left(\frac{x}{y}\right)^{k+1} < \left(\frac{x}{y}\right)^k \implies \left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^{k+1} < \left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^k \implies \left(\frac{x}{y}\right)^{k+2} < \left(\frac{x}{y}\right)^{k+1}.$$

Therefore, $\left(\frac{x}{y}\right)^{n+1} < \left(\frac{x}{y}\right)^n$ for all integers $n \geq 1$.

29. 1. When $n = 1$, $1 + a \geq a$ since $1 > 0$.

2. Assume $(1 + a)^k \geq ka$.

Then, $(1 + a)^{k+1} = (1 + a)^k(1 + a) \geq ka(1 + a)$

$$= ka + ka^2 \geq ka + a \quad \text{(because } a > 1\text{)}$$

$$= (k + 1)a.$$

Therefore, by mathematical induction, the inequality is valid for all integers $n \geq 1$.

30. 1. When $n = 1$, $3^1 > (1)2^1$

2. Assume that $3^k > k2^k$, $k \ge 2$.

First note that $k \ge 2 \Longrightarrow 3k \ge 2k + 2 = 2(k + 1)$

Then, $3^{k+1} = 3(3^k) > 3(k2^k) = (3k)2^k \ge 2(k + 1)2^k = (k + 1)2^{k+1}$.

Therefore $3^n > n2^n$ for all integers $n \ge 1$.

31. 1. When $n = 1$, $(ab)^1 = a^1b^1 = ab$.

2. Assume that $(ab)^k = a^kb^k$.

Then, $(ab)^{k+1} = (ab)^k(ab)$

$= a^kb^kab$

$= a^{k+1}b^{k+1}$.

Thus, $(ab)^n = a^nb^n$.

32. 1. When $n = 1$, $\left(\frac{a}{b}\right)^1 = \frac{a^1}{b^1}$.

2. Assume that $\left(\frac{a}{b}\right)^k = \frac{a^k}{b^k}$.

Then, $\left(\frac{a}{b}\right)^{k+1} = \left(\frac{a}{b}\right)^k\left(\frac{a}{b}\right) = \frac{a^k}{b^k} \cdot \frac{a}{b} = \frac{a^{k+1}}{b^{k+1}}$.

Thus, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

33. 1. When $n = 1$, $(x_1)^{-1} = x_1^{-1}$.

2. Assume that

$(x_1x_2x_3 \cdot \cdot \cdot x_k)^{-1} = x_1^{-1}x_2^{-1}x_3^{-1} \cdot \cdot \cdot x_k^{-1}$.

Then,

$$\begin{aligned}(x_1x_2x_3 \cdots x_kx_{k+1})^{-1} &= [(x_1x_2x_3 \cdots x_k)x_{k+1}]^{-1} \\ &= (x_1x_2x_3 \cdots x_k)^{-1}x_{k+1}^{-1} \\ &= x_1^{-1}x_2^{-1}x_3^{-1} \cdots x_k^{-1}x_{k+1}^{-1}.\end{aligned}$$

Thus, the formula is valid.

34. 1. When $n = 1$, $\ln x_1 = \ln x_1$.

2. Assume that $\ln(x_1x_2x_3 \ldots x_k) = \ln x_1 + \ln x_2 + \ln x_3 + \cdot \cdot \cdot + \ln x_k$.

$$\begin{aligned}\text{Then, } \ln(x_1x_2x_3 \ldots x_k x_{k+1}) &= \ln[(x_1x_2x_3 \ldots x_k)x_{k+1}] \\ &= \ln(x_1x_2x_3 \ldots x_k) + \ln x_{k+1} \\ &= \ln x_1 + \ln x_2 + \ln x_3 + \cdots + \ln x_k + \ln x_{k+1}.\end{aligned}$$

Thus, $\ln(x_1x_2x_3 \ldots x_n) = \ln x_1 + \ln x_2 + \ln x_3 + \cdot \cdot \cdot + \ln x_n$.

35. 1. When $n = 1$, $x(y_1) = xy_1$.

2. Assume that $x(y_1 + y_2 + \cdot \cdot \cdot + y_k) = xy_1 + xy_2 + \cdot \cdot \cdot + xy_k$.

Then,

$$\begin{aligned}xy_1 + xy_2 + \cdots + xy_k + xy_{k+1} &= x(y_1 + y_2 + \cdots + y_k) + xy_{k+1} \\ &= x[(y_1 + y_2 + \cdots + y_k) + y_{k+1}] \\ &= x(y_1 + y_2 + \cdots + y_k + y_{k+1}).\end{aligned}$$

Hence, the formula holds.

36. 1. When $n = 1$, $a + bi$ and $a - bi$ are complex conjugates by definition.

2. Assume that $(a + bi)^k$ and $(a - bi)^k$ are complex conjugates.

That is, if $(a + bi)^k = c + di$, then $(a - bi)^k = c - di$.

Then,

$$(a + bi)^{k+1} = (a + bi)^k(a + bi) = (c + di)(a + bi)$$
$$= (ac - bd) + i(bc + ad)$$
$$\text{and } (a - bi)^{k+1} = (a - bi)^k(a - bi) = (c - di)(a - bi)$$
$$= (ac - bd) - i(bc + ad).$$

This implies that $(a + bi)^{k+1}$ and $(a - bi)^{k+1}$ are complex conjugates.
Therefore, $(a + bi)^n$ and $(a - bi)^n$ are complex conjugates for $n \geq 1$.

37. 1. When $n = 1$, $[1^3 + 3(1)^2 + 2(1)] = 6$ and 3 is a factor.

2. Assume that 3 is a factor of $(k^3 + 3k^2 + 2k)$.

Then,

$$[(k + 1)^3 + 3(k + 1)^2 + 2(k + 1)] = k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 2k + 2$$
$$= (k^3 + 3k^2 + 2k) + (3k^2 + 9k + 6)$$
$$= (k^3 + 3k^2 + 2k) + 3(k^2 + 3k + 2).$$

Since 3 is a factor of $(k^3 + 3k^2 + 2k)$ by our assumption, and 3 is a factor of $3(k^2 + 3k + 2)$ then 3 is a factor of the whole sum.

Thus, 3 is a factor of $(n^3 + 3n^2 + 2n)$ for every positive integer n.

38. 1. When $n = 1$, 3 is a factor of $[1^3 + 5(1) + 6] = 12$.

2. Assume that 3 is a factor of $k^3 + 5k + 6$.

Then, $(k + 1)^3 + 5(k + 1) + 6 = k^3 + 3k^2 + 3k + 1 + 5k + 11$

$$= k^3 + 3k^2 + 8k + 12$$
$$= (k^3 + 5k + 6) + (3k^2 + 3k + 6)$$
$$= (k^3 + 5k + 6) + 3(k^2 + k + 2).$$

Because 3 is a factor of both terms, 3 is a factor of $(k + 1)^3 + 5(k + 1) + 6$.

Therefore, 3 is a factor of $n^3 + 5n + 6$ for all $n > 0$.

39. 1. When $n = 1$, $[1^3 - 1 + 3] = 3$, and 3 is a factor.

2. Assume that 3 is a factor of $k^3 - k + 3$. Then,

$$[(k + 1)^3 - (k + 1) + 3] = k^3 + 3k^2 + 3k + 1 - k - 1 + 3$$
$$= k^3 + 3k^2 + 2k + 3$$
$$= (k^3 - k + 3) + 3k^2 + 3k$$
$$= (k^3 - k + 3) + 3(k^2 + k).$$

Since 3 is a factor of $k^3 - k + 3$ by our assumption, and 3 is a factor of $3(k^2 + k)$, then 3 is a factor of the whole sum.

Thus, 3 is a factor of $n^3 - n + 3$ for every positive integer n.

40. 1. When $n = 1$, $[1^4 - 1 + 4] = 4$, and 2 is a factor.

2. Assume that 2 is a factor of $k^4 - k + 4$.

Then,

$$\begin{aligned}[(k+1)^4 - (k+1) + 4] &= (k^4 + 4k^3 + 6k^2 + 4k + 1) - k - 1 + 4\\ &= k^4 + 4k^3 + 6k^2 + 3k + 4\\ &= (k^4 - k + 4) + (4k^3 + 6k^2 + 4k).\end{aligned}$$

Since 2 is a factor of each term, it is a factor of the sum.

Thus, 2 is a factor of $n^4 - n + 4$ for each positive integer n.

41. 1. When $n = 1$, $2^{2+1} + 1 = 9$, and 3 is a factor.

2. Assume that 3 is a factor of $2^{2k+1} + 1$.

Then,

$$\begin{aligned}2^{2(k+1)+1} + 1 &= 2^{2k+3} + 1\\ &= 4 \cdot 2^{2k+1} + 1\\ &= (3+1)2^{2k+1} + 1\\ &= (2^{2k+1} + 1) + 3 \cdot 2^{2k+1}.\end{aligned}$$

Since 3 is a factor of $2^{2k+1} + 1$ by our assumption, and 3 is a factor of $3 \cdot 2^{2k+1}$, then 3 is a factor of the whole sum.

Thus, 3 is a factor of $2^{2n+1} + 1$ for every positive integer n.

42. 1. When $n = 1$, $2^{4(1)-2} + 1 = 5$, and 5 is a factor.

2. Assume that 5 is a factor of $2^{4k-2} + 1$.

Then,

$$\begin{aligned}2^{4(k+1)-2} + 1 &= 2^{4k+2} + 1\\ &= 16 \cdot 2^{4k-2} + 1\\ &= (15+1)2^{4k-2} + 1\\ &= (2^{4k-2} + 1) + 15 \cdot 2^{4k-2}.\end{aligned}$$

Since 5 is a factor of $2^{4k-2} + 1$ by our assumption, and 5 is a factor of $15 \cdot 2^{4k-2}$, then 5 is a factor of the whole sum.

Thus, 5 is a factor of $2^{4n-2} + 1$ for every positive integer n.

43. $a_1 = 0$, $a_n = a_{n-1} + 3$

$a_1 = 0$

$a_2 = a_1 + 3 = 0 + 3 = 3$

$a_3 = a_2 + 3 = 3 + 3 = 6$

$a_4 = a_3 + 3 = 6 + 3 = 9$

$a_5 = a_4 + 3 = 9 + 3 = 12$

a_n:	0		3		6		9		12
First differences:		3		3		3		3	
Second differences:			0		0		0		

Since the first differences are equal, the sequence has a linear model.

44. $a_1 = 2$, $a_n = n - a_{n-1}$

$a_1 = 2$

$a_2 = n - a_1 = 2 - 2 = 0$

$a_3 = n - a_2 = 3 - 0 = 3$

$a_4 = n - a_3 = 4 - 3 = 1$

$a_5 = n - a_5 = 5 - 1 = 4$

a_n:	2		0		3		1		4
First differences:		-2		3		-2		3	
Second differences:			5		-5		5		

Since neither the first differences nor the second differences are equal, the sequence does not have a linear or quadratic model.

45. $a_1 = 3, a_n = a_{n-1} - n$

$a_1 = 3$

$a_2 = a_1 - 2 = 3 - 2 = 1$

$a_3 = a_2 - 3 = 1 - 3 = -2$

$a_4 = a_3 - 4 = -2 - 4 = -6$

$a_5 = a_4 - 5 = -6 - 5 = -11$

a_n: 3 1 −2 −6 −11

First differences: −2 −3 −4 −5

Second differences: −1 −1 −1

Since the second differences are all the same, the sequence has a quadratic model.

46. $a_2 = -3, \quad a_n = -2a_{n-1}$

$a_2 = -3$

$a_3 = -2a_2 = -2(-3) = 6$

$a_4 = -2a_3 = -2(6) = -12$

$a_5 = -2a_4 = -2(-12) = 24$

$a_6 = -2a_5 = -2(24) = -48$

a_n: −3 6 −12 24 −48

First differences: 9 −18 36 −72

Second differences: −27 54 −108

Since neither the first nor the second differences are equal, the sequence does not have a linear or quadratic model.

47. $a_0 = 0, a_n = a_{n-1} + n$

$a_0 = 0$

$a_1 = a_0 + 1 = 0 + 1 = 1$

$a_2 = a_1 + 2 = 1 + 2 = 3$

$a_3 = a_2 + 3 = 3 + 3 = 6$

$a_4 = a_3 + 4 = 6 + 4 = 10$

a_n: 0 1 3 6 10

First differences: 1 2 3 4

Second differences: 1 1 1

Since the second differences are equal, the sequence has a quadratic model.

48. $a_0 = 2, \quad a_n = (a_{n-1})^2$

$a_0 = 2$

$a_1 = a_0^{\,2} = 2^2 = 4$

$a_2 = a_1^{\,2} = 4^2 = 16$

$a_3 = a_2^{\,2} = 16^2 = 256$

$a_4 = a_3^{\,2} = 256^2 = 65{,}536$

a_n: 2 4 16 256 65,536

First differences: 2 12 240 65,280

Second differences: 10 228 65,040

Since neither the first differences nor the second differences are equal, the sequence does not have a linear or quadratic model.

49. $a_1 = 2, a_n = a_{n-1} + 2$

$a_1 = 2$

$a_2 = a_1 + 2 = 2 + 2 = 4$

$a_3 = a_2 + 2 = 4 + 2 = 6$

$a_4 = a_3 + 2 = 6 + 2 = 8$

$a_5 = a_4 + 2 = 8 + 2 = 10$

a_n: 2 4 6 8 10

First differences: 2 2 2 2

Second differences: 0 0 0

Since the first differences are equal, the sequence has a linear model.

50. $a_1 = 0, \quad a_n = a_{n-1} + 2n$

$a_1 = 0$

$a_2 = a_1 + 2(2) = 0 + 4 = 4$

$a_3 = a_2 + 2(3) = 4 + 6 = 10$

$a_4 = a_3 + 2(4) = 10 + 8 = 18$

$a_5 = a_4 + 2(5) = 18 + 10 = 28$

a_n: 0 4 10 18 28

First differences: 4 6 8 10

Second differences: 2 2 2

Since the second differences are equal, the sequence has a quadratic model.

51. $a_1 = 3, a_2 = 3, a_3 = 5$

Let $a_n = an^2 + bn + c$.

$a_1 = a(1)^2 + b(1) + c = 3 \Rightarrow a + b + c = 3$

$a_2 = a(2)^2 + b(2) + c = 3 \Rightarrow 4a + 2b + c = 3$

$a_3 = a(3)^2 + b(3) + c = 5 \Rightarrow 9a + 3b + c = 5$

Solving the system, $a = 1, b = -3, c = 5$,
$a_n = n^2 - 3n + 5, n \ge 1$.

52. $a_1 = 7, a_2 = 6, a_3 = 7$

Let $a_n = an^2 + bn + c$.

$a_1 = a(1)^2 + b(1) + c = 7 \Rightarrow a + b + c = 7$

$a_2 = a(2)^2 + b(2) + c = 6 \Rightarrow 4a + 2b + c = 6$

$a_3 = a(3)^2 + b(3) + c = 7 \Rightarrow 9a + 3b + c = 7$

Solving the system, $a = 1, b = -4, c = 10$,
$a_n = n^2 - 4n + 10, n \ge 1$.

53. $a_0 = -3, a_2 = 1, a_4 = 9$

Let $a_n = an^2 + bn + c$. Then:

$$\begin{aligned}
a_0 &= a(0)^2 + b(0) + c = -3 \Rightarrow c = -3\\
a_2 &= a(2)^2 + b(2) + c = 1 \Rightarrow 4a + 2b + c = 1\\
&\qquad 4a + 2b = 4\\
&\qquad 2a + b = 2\\
a_4 &= a(4)^2 + b(4) + c = 9 \Rightarrow 16a + 4b + c = 9\\
&\qquad 16a + 4b = 12\\
&\qquad 4a + b = 3
\end{aligned}$$

By elimination:
$$\begin{aligned}
-2a - b &= -2\\
\underline{4a + b} &= \underline{3}\\
2a &= 1\\
a = \tfrac{1}{2} &\Rightarrow b = 1
\end{aligned}$$

Thus, $a_n = \frac{1}{2}n^2 + n - 3$.

54. $a_0 = 3, \quad a_2 = 0, \quad a_6 = 36$

Let $a_n = an^2 + bn + c$. Thus:

$$\begin{aligned}
a_0 &= a(0)^2 + b(0) + c = 3 \Rightarrow c = 3\\
a_2 &= a(2)^2 + b(2) + c = 0 \Rightarrow 4a + 2b + c = 0\\
&\qquad 4a + 2b = -3\\
a_6 &= a(6)^2 + b(6) + c = 36 \Rightarrow 36a + 6b + c = 36\\
&\qquad 36a + 6b = 33\\
&\qquad 12a + 2b = 11
\end{aligned}$$

By elimination:
$$\begin{aligned}
-4a - 2b &= 3\\
\underline{12a + 2b} &= \underline{11}\\
8a &= 14\\
a = \tfrac{7}{4} &\Rightarrow b = -5
\end{aligned}$$

Thus, $a_n = \frac{7}{4}n^2 - 5n + 3$.

55. (a) $n = 1$: 3 sides

$n = 2$: $3 \cdot 4 = 12$ sides

$n = 3$: $3 \cdot 4^2 = 48$ sides

nth Koch snowflake: $3(4)^{n-1}$ sides

To prove this, use mathematical induction.

1. For $n = 1$, the number of sides is $3 \cdot 4^{1-1} = 3$.
2. Assume that the number of sides of the kth Koch snowflake is $3 \cdot 4^{k-1}$. When the $(k+1)^{st}$ Koch snowflake is created, each side is replaced with 4 sides. That is, the number of sides is increased by a factor of 4:

   Number sides $= 4(3 \cdot 4^{k-1}) = 3 \cdot 4^k$.

Hence, the formula is valid for all positive integers n.

(b) $n = 1$: $A_1 = \dfrac{\sqrt{3}}{4}(1)^2 = \dfrac{\sqrt{3}}{4}$

$n = 2$: $A_2 = \dfrac{\sqrt{3}}{4}\left[1 + \dfrac{1}{3}\right]$

$n = 3$: $A_3 = \dfrac{\sqrt{3}}{4}\left[1 + \dfrac{1}{3} + \dfrac{1}{3}\left(\dfrac{4}{9}\right)\right]$

$n = 4$: $A_4 = \dfrac{\sqrt{3}}{4}\left[1 + \dfrac{1}{3} + \dfrac{1}{3}\left(\dfrac{4}{9}\right) + \dfrac{1}{3}\left(\dfrac{4}{9}\right)^2\right]$

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \sum_{k=2}^{n} \frac{1}{3}\left(\frac{4}{9}\right)^{k-2}\right],\ n > 1$$

(c) For the nth Koch snowflake, the length of a single side is $(1/3)^{n-1}$, and the number of sides is $3 \cdot 4^{n-1}$. Hence, the perimeter is

$$\left(\frac{1}{3}\right)^{n-1} 3 \cdot 4^{n-1} = 3\left(\frac{4}{3}\right)^{n-1}.$$

56. (a) One ring $\rightarrow$ one move

Two rings $\rightarrow$ three moves

Three rings $\rightarrow$ seven moves

(b) Four rings: 7 moves to move 3

1 move for fourth ring

7 moves to bring back 3

Total: 15 moves

(c) If n rings, let h_n be the number of moves. Then,

$h_1 = 1$

$h_n = 2h_{n-1} + 1 = 2^n - 1$ moves.

(d) 1. For one ring, $h_1 = 1$.

2. Assume that $h_n = 2h_{n-1} + 1$. For $n + 1$ rings, it takes h_n moves to move n rings, one to move the last ring, and h_n more to move the n rings back.

   Total: $h_{n+1} = 2(h_n) + 1$

57. False. P_1 might not even be defined.

58. False. See the Study Tip on page 550.

59. False. It has $n - 2$ second differences.

60. (a) If P_3 is true and P_k implies P_{k+1}, then P_n is true for integers $n \geq 3$.

(b) If $P_1, P_2, P_3, \ldots, P_{50}$ are all true, then P_n is true for integers $1 \leq n \leq 50$.

(c) If P_1, P_2, and P_3 are all true, but the truth of P_k does not imply that P_{k+1} is true, then you may only conclude that P_1, P_2, and P_3 are true.

(d) If P_2 is true and P_{2k} implies P_{2k+2}, then P_{2n} is true for any positive integer n.

61. $(2x^2 - 1)^2 = 4x^4 - 4x^2 + 1$

62. $(2x - y)^2 = 4x^2 - 4xy + y^2$

63. $(5 - 4x)^3 = -64x^3 + 240x^2 - 300x + 125$

64. $(2x - 4y)^3 = 8x^3 - 48x^2y + 96xy^2 - 64y^3$

65. $3\sqrt{-27} - \sqrt{-12} = 3\sqrt{3 \cdot 3 \cdot (-3)} - \sqrt{2 \cdot 2(-3)} = 9\sqrt{3}i - 2\sqrt{3}i = 7\sqrt{3}i$

66. $\sqrt[3]{125} + 4\sqrt[3]{-8} - 2\sqrt[3]{-54} = 5 + 4(-2) + 6\sqrt[3]{2} = -3 + 6\sqrt[3]{2}$

67. $10\left(\sqrt[3]{64} - 2\sqrt[3]{-16}\right) = 10\left(4 - 2^2\sqrt[3]{-2}\right)$
$= 40 - 40\sqrt[3]{-2}$
$= 40\left(1 + \sqrt[3]{2}\right)$

68. $\left(-5 + \sqrt{-9}\right)^2 = (-5 + 3i)^2$
$= 25 - 9 - 30i = 16 - 30i$

Section 8.5 The Binomial Theorem

- You should be able to use the Binomial Theorem

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + {}_nC_r x^{n-r}y^r + \cdots + y^n$$

where ${}_nC_r = \dfrac{n!}{(n-r)!r!}$, to expand $(x+y)^n$.

- You should be able to use Pascal's Triangle.

Vocabulary Check

1. binomial coefficients
2. Binomial Theorem, Pascal's Triangle
3. ${}_nC_r$ or $\binom{n}{r}$
4. expanding, binomial

1. ${}_7C_5 = \dfrac{7!}{2!5!} = \dfrac{7 \cdot 6 \cdot 5!}{2 \cdot 5!} = \dfrac{42}{2} = 21$

2. ${}_9C_6 = \dfrac{9!}{6!3!} = \dfrac{9 \cdot 8 \cdot 7 \cdot 6!}{6!\,3 \cdot 2} = \dfrac{9 \cdot 8 \cdot 7}{6} = 84$

3. $\binom{12}{0} = {}_{12}C_0 = \dfrac{12!}{0!12!} = 1$

4. $\binom{20}{20} = {}_{20}C_{20} = \dfrac{20!}{20!0!} = 1$

5. ${}_{20}C_{15} = \dfrac{20!}{15!5!} = \dfrac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 15{,}504$

6. ${}_{12}C_3 = \dfrac{12!}{9!3!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9!}{9!\,3 \cdot 2} = 220$

7. ${}_{14}C_1 = \dfrac{14!}{13!1!} = \dfrac{14 \cdot 13!}{13!} = 14$

8. ${}_{18}C_{17} = \dfrac{18!}{17!1!} = \dfrac{18 \cdot 17!}{17!} = 18$

9. $\binom{100}{98} = {}_{100}C_{98} = \dfrac{100!}{98!2!} = \dfrac{100 \cdot 99}{2 \cdot 1} = 4950$

10. $\binom{10}{7} = \dfrac{10!}{7!3!} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7!}{7!\,3 \cdot 2} = 120$

11. ${}_{41}C_{36} = 749{,}398$

12. ${}_{34}C_4 = 46{,}376$

13. ${}_{100}C_{98} = 4950$

14. ${}_{500}C_{498} = 124{,}750$

15. ${}_{250}C_2 = 31{,}125$

16. ${}_{1000}C_2 = 499{,}500$

17. $(x+2)^4 = {}_4C_0x^4 + {}_4C_1x^3(2) + {}_4C_2x^2(2)^2 + {}_4C_3x(2)^3 + {}_4C_4(2)^4$
$= x^4 + 8x^3 + 24x^2 + 32x + 16$

18. $(x+1)^6 = {}_6C_0x^6 + {}_6C_1x^5(1) + {}_6C_2x^4(1)^2 + {}_6C_3x^3(1)^3 + {}_6C_4x^2(1)^4 + {}_6C_5x(1)^5 + {}_6C_6(1)^6$
$= x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

19. $(a+3)^3 = {}_3C_0a^3 + {}_3C_1a^2(3) + {}_3C_2a(3)^2 + {}_3C_3(3)^3$
$= a^3 + 3a^2(3) + 3a(3)^2 + (3)^3 = a^3 + 9a^2 + 27a + 27$

20. $(a + 2)^4 = {}_4C_0a^4 + {}_4C_1a^3(2) + {}_4C_2a^2(2)^2 + {}_4C_3a(2)^3 + {}_4C_4(2)^4 = a^4 + 8a^3 + 24a^2 + 32a + 16$

21. $(y - 2)^4 = {}_4C_0y^4 - {}_4C_1y^3(2) + {}_4C_2y^2(2)^2 - {}_4C_3y(2)^3 + {}_4C_4(2)^4$

$= y^4 - 4y^3(2) + 6y^2(4) - 4y(8) + 16$

$= y^4 - 8y^3 + 24y^2 - 32y + 16$

22. $(y - 2)^5 = {}_5C_0y^5 - {}_5C_1y^4(2) + {}_5C_2y^3(2)^2 - {}_5C_3y^2(2)^3 + {}_5C_4y(2)^4 - {}_5C_5(2)^5$

$= y^5 - 10y^4 + 40y^3 - 80y^2 + 80y - 32$

23. $(x + y)^5 = {}_5C_0x^5 + {}_5C_1x^4y + {}_5C_2x^3y^2 + {}_5C_3x^2y^3 + {}_5C_4xy^4 + {}_5C_5y^5$

$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

24. $(x + y)^6 = {}_6C_0x^6 + {}_6C_1x^5y + {}_6C_2x^4y^2 + {}_6C_3x^3y^3 + {}_6C_4x^2y^4 + {}_6C_5xy^5 + {}_6C_6y^6$

$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

25. $(3r + 2s)^6 = {}_6C_0(3r)^6 + {}_6C_1(3r)^5(2s) + {}_6C_2(3r)^4(2s)^2 + {}_6C_3(3r)^3(2s)^3 + {}_6C_4(3r)^2(2s)^4 + {}_6C_5(3r)(2s)^5 + {}_6C_6(2s)^6$

$= 729r^6 + 2916r^5s + 4860r^4s^2 + 4320r^3s^3 + 2160r^2s^4 + 576rs^5 + 64s^6$

26. $(4x + 3y)^4 = 256x^4 + 768x^3y + 864x^2y^2 + 432xy^3 + 81y^4$

27. $(x - y)^5 = {}_5C_0x^5 - {}_5C_1x^4y + {}_5C_2x^3y^2 - {}_5C_3x^2y^3 + {}_5C_4xy^4 - {}_5C_5y^5$

$= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

28. $(2x - y)^5 = {}_5C_0(2x)^5 - {}_5C_1(2x)^4y + {}_5C_2(2x)^3y^2 - {}_5C_3(2x)^2y^3 + {}_5C_4(2x)y^4 - {}_5C_5y^5$

$= 32x^5 - 5(16x^4)y + 10(8x^3)y^2 - 10(4x^2)y^3 + 5(2x)y^4 - y^5$

$= 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$

29. $(1 - 4x)^3 = {}_3C_01^3 - {}_3C_11^2(4x) + {}_3C_21(4x)^2 - {}_3C_3(4x)^3$

$= 1 - 3(4x) + 3(4x)^2 - (4x)^3$

$= 1 - 12x + 48x^2 - 64x^3$

30. $(5 - 2y)^3 = 125 - 150y + 60y^2 - 8y^3$

31. $(x^2 + 2)^4 = {}_4C_0(x^2)^4 + {}_4C_1(x^2)^3(2) + {}_4C_2(x^2)^22^2 + {}_4C_3(x^2)2^3 + {}_4C_4(2)^4$

$= x^8 + 8x^6 + 24x^4 + 32x^2 + 16$

32. $(3 - y^2)^3 = {}_3C_0(3)^3 - {}_3C_1(3)^2y^2 + {}_3C_2(3)(y^2)^2 - {}_3C_3(y^2)^3$

$= 27 - 27y^2 + 9y^4 - y^6$

$= -y^6 + 9y^4 - 27y^2 + 27$

33. $(x^2 - 5)^5 = {}_5C_0(x^2)^5 - {}_5C_1(x^2)^4(5) + {}_5C_2(x^2)^3(5^2) - {}_5C_3(x^2)^2(5^3) + {}_5C_4(x^2)(5^4) - {}_5C_5(5^5)$

$= x^{10} - 25x^8 + 250x^6 - 1250x^4 + 3125x^2 - 3125$

34. $(y^2 + 1)^6 = {}_6C_0(y^2)^6 + {}_6C_1(y^2)^5 + {}_6C_2(y^2)^4 + {}_6C_3(y^2)^3 + {}_6C_4(y^2)^2 + {}_6C_5(y^2) + {}_6C_6$
$= y^{12} + 6y^{10} + 15y^8 + 20y^6 + 15y^4 + 6y^2 + 1$

35. $(x^2 + y^2)^4 = {}_4C_0(x^2)^4 + {}_4C_1(x^2)^3(y^2) + {}_4C_2(x^2)^2(y^2)^2 + {}_4C_3(x^2)(y^2)^3 + {}_4C_4(y^2)^4$
$= x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8$

36. $(x^2 + y^2)^6 = {}_6C_0(x^2)^6 + {}_6C_1(x^2)^5(y^2) + {}_6C_2(x^2)^4(y^2)^2 + {}_6C_3(x^2)^3(y^2)^3 + {}_6C_4(x^2)^2(y^2)^4 + {}_6C_5(x^2)(y^2)^5 + {}_6C_6(y^2)^6$
$= x^{12} + 6x^{10}y^2 + 15x^8y^4 + 20x^6y^6 + 15x^4y^8 + 6x^2y^{10} + y^{12}$

37. $(x^3 - y)^6 = {}_6C_0(x^3)^6 - 6C_1(x^3)^5 y + {}_6C_2(x^3)^4y^2 - {}_6C_3(x^3)^3 y^3 + {}_6C_4(x^3)^2y^4 - {}_6C_5(x^3)y^5 + {}_6C_6\, y^6$
$= x^{18} - 6x^{15}y + 15x^{12}y^2 - 20x^9y^3 + 15x^6y^4 - 6x^3y^5 + y^6$

38. $(2x^3 - y)^5 = {}_5C_0(2x^3)^5 - {}_5C_1(2x^3)^4y + {}_5C_2(2x^3)^3y^2 - {}_5C_3(2x^3)^2y^3 + {}_5C_4(2x^3)y^4 - {}_5C_5y^5$
$= 32x^{15} - 80x^{12}y + 80x^9y^2 - 40x^6y^3 + 10x^3y^4 - y^5$

39. $\left(\frac{1}{x} + y\right)^5 = {}_5C_0\left(\frac{1}{x}\right)^5 + {}_5C_1\left(\frac{1}{x}\right)^4 y + {}_5C_2\left(\frac{1}{x}\right)^3 y^2 + {}_5C_3\left(\frac{1}{x}\right)^2 y^3 + {}_5C_4\left(\frac{1}{x}\right)y^4 + {}_5C_5y^5$
$= \frac{1}{x^5} + \frac{5y}{x^4} + \frac{10y^2}{x^3} + \frac{10y^3}{x^2} + \frac{5y^4}{x} + y^5$

40. $\left(\frac{1}{x} + 2y\right)^6 = {}_6C_0\left(\frac{1}{x}\right)^6 + {}_6C_1\left(\frac{1}{x}\right)^5(2y) + {}_6C_2\left(\frac{1}{x}\right)^4(2y)^2 + {}_6C_3\left(\frac{1}{x}\right)^3(2y)^3 + {}_6C_4\left(\frac{1}{x}\right)^2(2y)^4 + {}_6C_5\left(\frac{1}{x}\right)(2y)^5 + {}_6C_6(2y)^6$
$= 1\left(\frac{1}{x}\right)^6 + 6(2)\left(\frac{1}{x}\right)^5 y + 15(4)\left(\frac{1}{x}\right)^4 y^2 + 20(8)\left(\frac{1}{x}\right)^3 y^3 + 15(16)\left(\frac{1}{x}\right)^2 y^4 + 6(32)\left(\frac{1}{x}\right)y^5 + 1(64)y^6$
$= \frac{1}{x^6} + \frac{12y}{x^5} + \frac{60y^2}{x^4} + \frac{160y^3}{x^3} + \frac{240y^4}{x^2} + \frac{192y^5}{x} + 64y^6$

41. $\left(\frac{2}{x} - y\right)^4 = {}_4C_0\left(\frac{2}{x}\right)^4 - {}_4C_1\left(\frac{2}{x}\right)^3 y + {}_4C_2\left(\frac{2}{x}\right)^2 y^2 - {}_4C_3\left(\frac{2}{x}\right)y^3 + {}_4C_4y^4$
$= \frac{16}{x^4} - \frac{32}{x^3}y + \frac{24}{x^2}y^2 - \frac{8}{x}y^3 + y^4$

42. $\left(\frac{2}{x} - 3y\right)^5 = {}_5C_0\left(\frac{2}{x}\right)^5 - {}_5C_1\left(\frac{2}{x}\right)^4(3y) + {}_5C_2\left(\frac{2}{x}\right)^3(3y)^2 - {}_5C_3\left(\frac{2}{x}\right)^2(3y)^3 + {}_5C_4\left(\frac{2}{x}\right)(3y)^4 - {}_5C_5(3y)^5$
$= \frac{32}{x^5} - \frac{240}{x^4}y + \frac{720}{x^3}y^2 - \frac{1080}{x^2}y^3 + \frac{810}{x}y^4 - 243y^5$

43. $(4x - 1)^3 - 2(4x - 1)^4 = (64x^3 - 48x^2 + 12x - 1) - 2(256x^4 - 256x^3 + 96x^2 - 16x + 1)$
$= -512x^4 + 576x^3 - 240x^2 + 44x - 3$

44. $(x + 3)^5 - 4(x + 3)^4 = (x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243) - 4(x^4 + 12x^3 + 54x^2 + 108x + 81)$
$= x^5 + 11x^4 + 42x^3 + 54x^2 - 27x - 81$

45. $2(x-3)^4 + 5(x-3)^2 = 2[x^4 - 4(x^3)(3) + 6(x^2)(3^2) - 4(x)(3^3) + 3^4] + 5[x^2 - 2(x)(3) + 3^2]$
$= 2(x^4 - 12x^3 + 54x^2 - 108x + 81) + 5(x^2 - 6x + 9)$
$= 2x^4 - 24x^3 + 113x^2 - 246x + 207$

46. $3(x+1)^5 + 4(x+1)^3 = (3x^5 + 15x^4 + 30x^3 + 30x^2 + 15x + 3) + (4x^3 + 12x^2 + 12x + 4)$
$= 3x^5 + 15x^4 + 34x^3 + 42x^2 + 27x + 7$

47. $-3(x-2)^3 - 4(x+1)^6 = [-3x^3 + 18x^2 - 36x + 24] - [4x^6 + 24x^5 + 60x^4 + 80x^3 + 60x^2 + 24x + 4]$
$= -4x^6 - 24x^5 - 60x^4 - 83x^3 - 42x^2 - 60x + 20$

48. $5(x+2)^5 - 2(x-1)^2 = [5x^5 + 50x^4 + 200x^3 + 400x^2 + 400x + 160] - [2x^2 - 4x + 2]$
$= 5x^5 + 50x^4 + 200x^3 + 398x^2 + 404x + 158$

49. $(x+8)^{10}, n = 4$
${}_{10}C_3 x^{10-3}(8)^3 = 120x^7(512) = 61{,}440x^7$

50. $(x-5)^6, \quad n = 7$
${}_6C_6\, x^0(-5)^6 = 15{,}625$

51. $(x-6y)^5, n = 3$
${}_5C_2 x^{5-2}(-6y)^2 = 10x^3(36)y^2 = 360x^3y^2$

52. $(x-10z)^7, \quad n = 4$
${}_7C_3\, x^{7-3}(-10z)^3 = -35{,}000\, x^4z^3$

53. $(4x+3y)^9, n = 8$
${}_9C_7(4x)^{9-7}(3y)^7 = 36(16)x^2(3^7)y^7$
$= 1{,}259{,}712x^2y^7$

54. $(5a+6b)^5, \quad n = 5$
${}_5C_4(5a)^{5-4}(6b)^4 = 32{,}400\, a \cdot b^4$

55. $(10x-3y)^{12}, n = 9$
${}_{12}C_8(10x)^{12-8}(-3y)^8 = 495(10^4)(3^8)x^4y^8$
$= 32{,}476{,}950{,}000x^4y^8$

56. $(7x+2y)^{15}, \quad n = 8$
${}_{15}C_7(7x)^{15-7}(2y)^7 \approx 4.7 \times 10^{12}\, x^8y^7$

57. The term involving x^4 in the expansion of $(x+3)^{12}$ is ${}_{12}C_8x^4(3)^8 = 495x^4(3)^8 = 3{,}247{,}695x^4$.
The coefficient is 3,247,695.

58. $(x+4)^{12}, \quad ax^5$
${}_{12}C_7\, x^5(4)^7 = 12{,}976{,}128x^5$
$a = 12{,}976{,}128$

59. The term involving x^8y^2 in the expansion of $(x-2y)^{10}$ is

$${}_{10}C_2x^8(-2y)^2 = \frac{10!}{2!8!} \cdot 4x^8y^2 = 180x^8y^2.$$

The coefficient is 180.

60. The term involving x^2y^8 in the expansion of $(4x-y)^{10}$ is

$${}_{10}C_8(4x)^2(-y)^8 = \frac{10!}{(10-8)!8!} \cdot 16x^2y^8 = 720x^2y^8.$$

The coefficient is 720.

61. The term involving x^6y^3 in $(3x - 2y)^9$ is

$${}_9C_3(3x)^6(-2y)^3 = 84(3)^6(-2)^3x^6y^3$$
$$= -489{,}888x^6y^3.$$

The coefficient is $-489{,}888$.

62. The term involving x^4y^4 in the expansion of $(2x - 3y)^8$ is

$${}_8C_4(2x)^4(-3y)^4 = 70(2^4)(-3)^4x^4y^4$$
$$= 90{,}720x^4y^4.$$

$a = 90{,}720$

63. The coefficient of $x^8y^6 = (x^2)^4y^6$ in the expansion of $(x^2 + y)^{10}$ is ${}_{10}C_6 = 210$.

64. The term involving z^6 in the expansion of $(z^2 - 1)^{12}$ is ${}_{12}C_9(z^2)^3(-1)^9 = \dfrac{12}{(12-9)!9!}z^6(-1) = -220z^6$.

The coefficient is -220.

65. 5th entry of 7th row: ${}_7C_5 = 21$

66. 3rd entry of 6th row: ${}_6C_3 = 20$

67. 5th entry of 6th row: ${}_6C_5 = 6$

68. 2nd entry of 5th row: ${}_5C_2 = 10$

69. 4th row of Pascal's Triangle: 1 4 6 4 1

$$(3t - 2v)^4 = 1(3t)^4 - 4(3t)^3(2v) + 6(3t)^2(2v)^2 - 4(3t)(2v)^3 + 1(2v)^4$$
$$= 81t^4 - 216t^3v + 216t^2v^2 - 96tv^3 + 16v^4$$

70. 4th row of Pascal's Triangle: 1 4 6 4 1

$$(5v - 2z)^4 = 1(5v)^4 - 4(5v)^3(2z) + 6(5v)^2(2z)^2 - 4(5v)(2z)^3 + 1(2z)^4$$
$$= 625v^4 - 1000v^3z + 600v^2z^2 - 160vz^3 + 16z^4$$

71. 5th row of Pascal's Triangle: 1 5 10 10 5 1

$$(2x - 3y)^5 = 1(2x)^5 - 5(2x)^4(3y) + 10(2x)^3(3y)^2 - 10(2x)^2(3y)^3 + 5(2x)(3y)^4 - (3y)^5$$
$$= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$$

72. 5th row of Pascal's Triangle: 1 5 10 10 5 1

$$(5y + 2)^5 = 1(5y)^5 + 5(5y)^42 + 10(5y)^32^2 + 10(5y)^22^3 + 5(5y)2^4 + 2^5$$
$$= 3125y^5 + 6250y^4 + 5000y^3 + 2000y^2 + 400y + 32$$

73. $$\left(\sqrt{x} + 5\right)^4 = \left(\sqrt{x}\right)^4 + 4\left(\sqrt{x}\right)^3(5) + 6\left(\sqrt{x}\right)^2(5)^2 + 4\left(\sqrt{x}\right)(5^3) + 5^4$$
$$= x^2 + 20x\sqrt{x} + 150x + 500\sqrt{x} + 625$$
$$= x^2 + 20x^{3/2} + 150x + 500x^{1/2} + 625$$

74. $$\left(4\sqrt{t} - 1\right)^3 = \left(4\sqrt{t}\right)^3 + 3\left(4\sqrt{t}\right)^2(-1) + 3\left(4\sqrt{t}\right)(-1)^2 + (-1)^3$$
$$= 64t\sqrt{t} - 48t + 12\sqrt{t} - 1 = 64t^{3/2} - 48t + 12t^{1/2} - 1$$

75. $$(x^{2/3} - y^{1/3})^3 = (x^{2/3})^3 - 3(x^{2/3})^2(y^{1/3}) + 3(x^{2/3})(y^{1/3})^2 - (y^{1/3})^3$$
$$= x^2 - 3x^{4/3}y^{1/3} + 3x^{2/3}y^{2/3} - y$$

76. $$(u^{3/5} + v^{1/5})^5 = u^3 + 5u^{12/5}v^{1/5} + 10u^{9/5}v^{2/5} + 10u^{6/5}v^{3/5} + 5u^{3/5}v^{4/5} + v$$

77. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{(x+h)^3 - x^3}{h}$

$= \dfrac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \dfrac{h(3x^2 + 3xh + h^2)}{h}$

$= 3x^2 + 3xh + h^2,\ h \neq 0$

78. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{(x+h)^4 - x^4}{h}$

$= \dfrac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$

$= \dfrac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h}$

$= 4x^3 + 6x^2h + 4xh^2 + h^3,\ h \neq 0$

79. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{(x+h)^6 - x^6}{h}$

$= \dfrac{(x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6) - x^6}{h}$

$= \dfrac{h(6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5)}{h}$

$= 6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5,\ h \neq 0$

80. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{(x+h)^8 - x^8}{h}$

$= \dfrac{(x^8 + 8x^7h + 28x^6h^2 + 56x^5h^3 + 70x^4h^4 + 56x^3h^5 + 28x^2h^6 + 8xh^7 + h^8) - x^8}{h}$

$= \dfrac{h(8x^7 + 28x^6h + 56x^5h^2 + 70x^4h^3 + 56x^3h^4 + 28x^2h^5 + 8xh^6 + h^7)}{h}$

$= 8x^7 + 28x^6h + 56x^5h^2 + 70x^4h^3 + 56x^3h^4 + 28x^2h^5 + 8xh^6 + h^7,\ h \neq 0$

81. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

$= \dfrac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$

$= \dfrac{(x+h) - x}{h\left[\sqrt{x+h} + \sqrt{x}\right]}$

$= \dfrac{1}{\sqrt{x+h} + \sqrt{x}},\ h \neq 0$

82. $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}$

$= \dfrac{\dfrac{x - (x+h)}{x(x+h)}}{h}$

$= \dfrac{\dfrac{-h}{x(x+h)}}{h}$

$= -\dfrac{1}{x(x+h)},\ h \neq 0$

83. $(1 + i)^4 = {}_4C_0 1^4 + {}_4C_1(1)^3 i + {}_4C_2(1)^2 i^2 + {}_4C_3 1 \cdot i^3 + {}_4C_4 i^4$

$= 1 + 4i - 6 - 4i + 1$

$= -4$

84. $(4 - i)^5 = 1024 - 1280i + 640i^2 - 160i^3 + 20i^4 - i^5$

$= 1024 - 1280i - 640 + 160i + 20 - i$

$= 404 - 1121i$

85. $(4 + i)^4 = {}_4C_0(4)^4 + {}_4C_1(4^3)i + {}_4C_2(4^2)(i^2) + {}_4C_3(4)(i^3) + {}_4C_4 i^4$

$= 256 + 256i - 96 - 16i + 1$

$= 161 + 240i$

86. $(2 - i)^5 = 2^5 - 5(2^4)i + 10(2^3)(i^2) - 10(2^2)(i^3) + 5(2)(i^4) - i^5$

$= 32 - 80i - 80 + 40i + 10 - i$

$= -38 - 41i$

87. $(2 - 3i)^6 = {}_6C_0 2^6 - {}_6C_1 2^5(3i) + {}_6C_2 2^4(3i)^2 - {}_6C_3 2^3(3i)^3 + {}_6C_4 2^2(3i)^4 - {}_6C_5 2(3i)^5 + {}_6C_6(3i)^6$

$= 64 - 576i - 2160 + 4320i + 4860 - 2916i - 729$

$= 2035 + 828i$

88. $(3 - 2i)^6 = 3^6 - 6(3^5)(2i) + 15(3^4)(2i)^2 - 20(3^3)(2i)^3 + 15(3^2)(2i)^4 - 6(3)(2i)^5 + (2i)^6$

$= 729 - 2916i - 4860 + 4320i + 2160 - 576i - 64$

$= -2035 + 828i$

89. $\left(5 + \sqrt{-16}\right)^3 = (5 + 4i)^3$

$= 5^3 + 3(5^2)(4i) + 3(5)(4i)^2 + (4i)^3$

$= 125 + 300i - 240 - 64i$

$= -115 + 236i$

90. $\left(5 + \sqrt{-9}\right)^3 = (5 + 3i)^3$

$= 5^3 + 3 \cdot 5^2(3i) + 3 \cdot 5(3i)^2 + (3i)^3$

$= 125 + 225i - 135 - 27i$

$= -10 + 198i$

91. $\left(4 + \sqrt{3}i\right)^4 = 4^4 + 4(4^3)\left(\sqrt{3}i\right) + 6(4^2)\left(\sqrt{3}i\right)^2 + 4(4)\left(\sqrt{3}i\right)^3 + \left(\sqrt{3}\right)^4$

$= 256 + 256\sqrt{3}i - 288 - 48\sqrt{3}i + 9$

$= -23 + 208\sqrt{3}i$

92. $\left(5 - \sqrt{3}i\right)^4 = 5^4 - 4 \cdot 5^3\left(\sqrt{3}i\right) + 6 \cdot 5^2\left(\sqrt{3}i\right)^2 - 4 \cdot 5\left(\sqrt{3}i\right)^3 + \left(\sqrt{3}i\right)^4$

$= 625 - 500\sqrt{3}i - 450 + 60\sqrt{3}i + 9$

$= 184 - 440\sqrt{3}i$

93. $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 = \frac{1}{8}(-1+\sqrt{3}i)^3$

$$= \frac{1}{8}\left[(-1)^3 + 3(-1)^2(\sqrt{3}i) + 3(-1)(\sqrt{3}i)^2 + (\sqrt{3}i)^3\right]$$

$$= \frac{1}{8}\left[-1 + 3\sqrt{3}i + 9 - 3\sqrt{3}i\right]$$

$$= 1$$

94. $\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3 = \left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}i\right) + 3\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right)^2 - \left(\frac{\sqrt{3}}{2}i\right)^3$

$$= \frac{1}{8} - \frac{3\sqrt{3}}{8}i - \frac{9}{8} + \frac{3\sqrt{3}}{8}i = -1$$

95. $\left(\frac{1}{4}-\frac{\sqrt{3}}{4}i\right)^3 = \left(\frac{1}{4}\right)^3 - 3\left(\frac{1}{4}\right)^2\left(\frac{\sqrt{3}}{4}i\right) + 3\left(\frac{1}{4}\right)\left(\frac{-\sqrt{3}}{4}i\right)^2 - \left(\frac{\sqrt{3}}{4}i\right)^3$

$$= \left[\frac{1}{64} - \frac{3}{4}\left(\frac{3}{16}\right)\right] + \left[\frac{-3}{16}\frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{64}\right]i$$

$$= -\frac{1}{8}$$

96. $\left(\frac{1}{3}-\frac{\sqrt{3}}{3}i\right)^3 = \left(\frac{1}{3}\right)^3 - 3\left(\frac{1}{3}\right)^2\left(\frac{\sqrt{3}i}{3}\right) + 3\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{3}i\right)^2 - \left(\frac{\sqrt{3}}{3}i\right)^3$

$$= \frac{1}{27} - \frac{\sqrt{3}}{9}i - \frac{1}{3} + \frac{\sqrt{3}}{9}i = -\frac{8}{27}$$

97. $(1.02)^8 = (1+0.02)^8 = 1 + 8(0.02) + 28(0.02)^2 + 56(0.02)^3 + 70(0.02)^4 + 56(0.02)^5$

$$+ 28(0.02)^6 + 8(0.02)^7 + (0.02)^8$$

$$= 1 + 0.16 + 0.0112 + 0.000448 + \cdots \approx 1.172$$

98. $(2.005)^{10} = (2+0.005)^{10} = 2^{10} + 10(2)^9(0.005) + 45(2)^8(0.005)^2 + 120(2)^7(0.005)^3 + 210(2)^6(0.005)^4$

$$+ 252(2)^5(0.005)^5 + 210(2)^4(0.005)^6 + 120(2)^3(0.005)^7 + 45(2)^2(0.005)^8$$

$$+ 10(2)(0.005)^9 + (0.005)^{10}$$

$$= 1024 + 25.6 + 0.288 + 0.00192 + 0.0000084 + \cdots$$

$$\approx 1049.890$$

99. $(2.99)^{12} = (3-0.01)^{12}$

$$= 3^{12} - 12(3)^{11}(0.01) + 66(3)^{10}(0.01)^2 - 220(3)^9(0.01)^3 + 495(3)^8(0.01)^4$$

$$- 792(3)^7(0.01)^5 + 924(3)^6(0.01)^6 - 792(3)^5(0.01)^7 + 495(3)^4(0.01)^8$$

$$- 220(3)^3(0.01)^9 + 66(3)^2(0.01)^{10} - 12(3)(0.01)^{11} + (0.01)^{12}$$

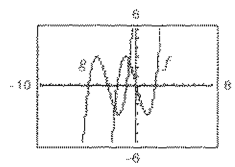
$$\approx 510{,}568.785$$

100. $$\begin{aligned}(1.98)^9 = (2 - 0.02)^9 &= 2^9 - 9(2)^8(0.02) + 36(2)^7(0.02)^2 - 84(2)^6(0.02)^3 + 126(2)^5(0.02)^4 \\ &\quad - 126(2)^4(0.02)^5 + 84(2)^3(0.02)^6 - 36(2)^2(0.02)^7 + 9(2)(0.02)^8 - (0.02)^9 \\ &= 512 - 46.08 + 1.8432 - 0.043008 + 0.00064512 \\ &\approx 467.721\end{aligned}$$

101. $f(x) = x^3 - 4x$

$$\begin{aligned}g(x) &= f(x + 3) \\ &= (x + 3)^3 - 4(x + 3) \\ &= x^3 + 9x^2 + 27x + 27 - 4x - 12 \\ &= x^3 + 9x^2 + 23x + 15\end{aligned}$$

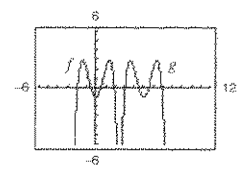
g is shifted three units to the left.



102. $f(x) = -x^4 + 4x^2 - 1$

$$\begin{aligned}g(x) &= f(x - 5) \\ &= -(x - 5)^4 + 4(x - 5)^2 - 1 \\ &= -(x^4 - 20x^3 + 150x^2 - 500x + 625) \\ &\quad + 4(x^2 - 10x + 25) - 1 \\ &= -x^4 + 20x^3 - 146x^2 + 460x - 526\end{aligned}$$

g is shifted five units to the right of f.



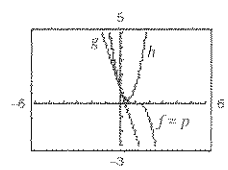
103. $f(x) = (1 - x)^3$

$g(x) = 1 - 3x$

$h(x) = 1 - 3x + 3x^2$

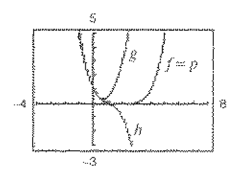
$p(x) = 1 - 3x + 3x^2 - x^3$

Since $p(x)$ is the expansion of $f(x)$, they have the same graph.



104. $p(x) = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4 = f(x)$

$p(x)$ is the expansion of $f(x)$.



105. ${}_7C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^3 = 35\left(\frac{1}{16}\right)\left(\frac{1}{8}\right) \approx 0.273$

106. ${}_{10}C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^7 = 120\left(\frac{1}{64}\right)\left(\frac{2187}{16,384}\right) \approx 0.2503$

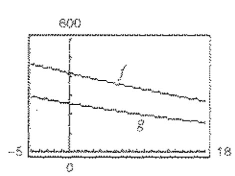
107. ${}_8C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^4 = 70\left(\frac{1}{81}\right)\left(\frac{16}{81}\right) \approx 0.171$

108. ${}_8C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^4 = 70\left(\frac{1}{16}\right)\left(\frac{1}{16}\right) \approx 0.2734$

109. $f(t) = 0.064t^2 - 9.30t + 416.5, 5 \le t \le 23$

(a) $$\begin{aligned}g(t) &= f(t + 20) \\ &= 0.064(t + 20)^2 - 9.30(t + 20) + 416.5 \\ &= 0.064t^2 - 6.74t + 256.1, -15 \le t \le 3\end{aligned}$$

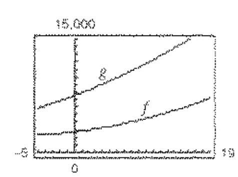
(b)



110. $f(t) = 6.22t^2 + 115.2t + 2730, 5 \le t \le 25$

(a) $$\begin{aligned}g(t) &= f(t + 20) \\ &= 6.22(t + 20)^2 + 115.2(t + 20) + 2730 \\ &= 6.22t^2 + 364t + 7522, -15 \le t \le 5\end{aligned}$$

(b)



111. False. The x^4y^8 term is

${}_{12}C_4x^4(-2y)^8 = 495x^4(-2)^8y^8 = 126,720x^4y^8$.

[**Note:** 7920 is the coefficient of x^8y^4.]

112. False. The coefficient of x^{10} is 1,732,104 and the coefficient of x^{14} is 192,456.

113. Answers will vary. See page 557.

114. Rows 8–10 of Pascal's Triangle are:

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

115. The expansions of $(x + y)^n$ and $(x - y)^n$ are almost the same except that the signs of the terms in the expansion of $(x - y)^n$ alternate from positive to negative.

116. (a) Second term of $(2x - 3y)^5$ is

$$5(2x)^4(-3y)^1 = -240x^4y.$$

(b) Fourth term of $\left(\frac{1}{2}x + 7y\right)^6$ is

$${}_6C_3\left(\tfrac{1}{2}x\right)^3(7y)^3 = 857.5x^3y^3.$$

117. $${}_nC_{n-r} = \frac{n!}{[n-(n-r)]!(n-r)!}$$
$$= \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = {}_nC_r$$

118. $$0 = (1-1)^n$$
$$= {}_nC_0 - {}_nC_1 + {}_nC_2 - {}_nC_3 + \cdots (\pm {}_nC_n)$$
$$= 0$$

119. $${}_nC_r + {}_nC_{r-1} = \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!}$$
$$= \frac{n!(n-r+1)}{(n-r)!r!(n-r+1)} + \frac{n!}{(n-r+1)!(r-1)!}\frac{r}{r}$$
$$= \frac{n!(n-r+1)}{(n-r+1)!r!} + \frac{n!r}{(n-r+1)!r!}$$
$$= \frac{n!(n-r+1+r)}{(n-r+1)!r!}$$
$$= \frac{n!(n+1)}{(n-r+1)!r!}$$
$$= \frac{(n+1)!}{(n+1-r)!r!} = {}_{n+1}C_r$$

120. ${}_nC_0 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \cdots + {}_nC_n = (1+1)^n = 2^n$

121. $g(x) = f(x) + 8$

$g(x)$ is shifted eight units up from $f(x)$.

122. $g(x) = f(x - 3)$

$g(x)$ is shifted three units to the right of $f(x)$.

123. $g(x) = f(-x)$

$g(x)$ is the reflection of $f(x)$ in the y-axis.

124. $g(x) = -f(x)$

$g(x)$ is the reflection of $f(x)$ in the x-axis.

125. $$\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}^{-1} = \frac{1}{-24+25}\begin{bmatrix} 4 & -5 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 5 & -6 \end{bmatrix}$$

126. $$\begin{bmatrix} 1.2 & -2.3 \\ -2 & 4 \end{bmatrix}^{-1} = \frac{1}{4.8-4.6}\begin{bmatrix} 4 & 2.3 \\ 2 & 1.2 \end{bmatrix} = \begin{bmatrix} 20 & 11.5 \\ 10 & 6 \end{bmatrix}$$

Section 8.6 Counting Principles

- You should know The Fundamental Counting Principle.
- ${}_nP_r = \dfrac{n!}{(n-r)!}$ is the number of permutations of n elements taken r at a time.
- Given a set of n objects that has n_1 of one kind, n_2 of a second kind, and so on, the number of distinguishable permutations is
$$\frac{n!}{n_1!n_2! \cdot \cdot \cdot n_k!}.$$
- ${}_nC_r = \dfrac{n!}{(n-r)!r!}$ is the number of combinations of n elements taken r at a time.

Vocabulary Check

1. Fundamental Counting Principle
2. permutation
3. ${}_nP_r = \dfrac{n!}{(n-r)!}$
4. distinguishable permutations
5. combinations

1. Odd integers: 1, 3, 5, 7, 9, 11

6 ways

2. Even integers: 2, 4, 6, 8, 10, 12

6 ways

3. Prime integers: 2, 3, 5, 7, 11

5 ways

4. Greater than 6: 7, 8, 9, 10, 11, 12

6 ways

5. Divisible by 4: 4, 8, 12

3 ways

6. Divisible by 3: 3, 6, 9, 12

4 ways

7. Sum is 8:

$1 + 7, 2 + 6, 3 + 5, 4 + 4, 5 + 3, 6 + 2, 7 + 1$

7 ways

8. Distinct integers whose sum is 8:

$1 + 7, 2 + 6, 3 + 5, 5 + 3, 6 + 2, 7 + 1$

6 ways

9. Amplifiers: 4 choices

Compact disc players: 6 choices

Speakers: 5 choices

Total: $4 \cdot 6 \cdot 5 = 120$ ways

10. Math courses: 2

Science courses: 3

Social sciences and humanities courses: 5

Total: $2 \cdot 3 \cdot 5 = 30$ ways

11. $2^{10} = 1024$ ways

12. First lock: $10 \cdot 10 \cdot 10$

Second lock: $10 \cdot 10 \cdot 10$

Hence,
$10^6 = 1{,}000{,}000$ combinations.

13. (a) $9 \cdot 10 \cdot 10 = 900$

(b) $9 \cdot 9 \cdot 8 = 648$

14. (a) $4 \cdot 10 \cdot 10 \cdot 10 = 4000$

(b) $9 \cdot 10 \cdot 10 \cdot 5 = 4500$

15. $2(8 \cdot 10 \cdot 10)(10 \cdot 10 \cdot 10 \cdot 10) = 16{,}000{,}000$ numbers

16. $4(8{,}000{,}000) = 32{,}000{,}000$ telephone numbers

17. (a) $26^3 + 26^3 = 35{,}152$

(b) There are $2 \cdot 25^3$ possibilities that don't have Q. Hence, $2 \cdot 26^3 - 2 \cdot 25^3 = 3902$ have at least one Q.

18. (a) $10^4 = 10{,}000$ ATM codes

(b) $9 \cdot 10^3 = 9000$ ATM codes that don't begin with zero

19. (a) $10^5 = 100{,}000$ zip codes

(b) $2 \cdot 10^4 = 20{,}000$ zip codes beginning with a one or a two

20. (a) 10^9 nine-digit zip codes

(b) $2 \cdot 10^8$ nine-digit zip codes beginning with a one or a two

21. (a) $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

(b) $6 \cdot 1 \cdot 4 \cdot 1 \cdot 2 \cdot 1 = 48$

22. (a) $8! = 40{,}320$ ways

(b) $(5!)(3!) = 120(6) = 720$ ways

23. $_nP_r = \dfrac{n!}{(n-r)!}$

So, $_4P_4 = \dfrac{4!}{0!} = 4! = 24.$

24. $_nP_r = \dfrac{n!}{(n-r)!}$

$_5P_5 = \dfrac{5!}{(5-5)!} = \dfrac{5!}{0!} = 120$

25. $_8P_3 = \dfrac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336$

26. $_{20}P_2 = \dfrac{20!}{18!} = 20(19) = 380$

27. $_5P_4 = \dfrac{5!}{1!} = 120$

28. $_7P_4 = \dfrac{7!}{3!}$

$= 7 \cdot 6 \cdot 5 \cdot 4 = 840$

29. $_{20}P_6 = 27{,}907{,}200$

30. $_{10}P_8 = 1{,}814{,}400$

31. $_{120}P_4 = 197{,}149{,}680$

32. $_{100}P_5 = 9{,}034{,}502{,}400$

33. $5! = 120$ ways

34. $4! = 24$

35. $9! = 362{,}880$ ways

36. $4! = 24$ ways

37. $_{12}P_4 = \dfrac{12!}{8!}$

$= 12 \cdot 11 \cdot 10 \cdot 9$

$= 11{,}880$ ways

38. $_{15}P_9 = \dfrac{15!}{6!}$

$= 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

$= 1{,}816{,}214{,}400$ ways

39. $37 \cdot 37 \cdot 37 = 50{,}653$

40. $_8P_3 = \dfrac{8!}{5!} = 336$ orders

41.

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

42. ABCD

ACBD

DBCA

DCBA

43. $\dfrac{7!}{2!1!3!1!} = \dfrac{7!}{2!3!} = 420$

44. $\dfrac{8!}{3!5!} = 56$

45. $\dfrac{7!}{2!1!1!1!1!1!} = \dfrac{7!}{2!}$

$= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$

$= 2520$

46. $\dfrac{11!}{1!4!4!2!} = \dfrac{11!}{4!4!2!} = 34{,}650$

47. ${}_5C_2 = \dfrac{5!}{2!3!} = \dfrac{5 \cdot 4}{2} = 10$

48. ${}_6C_3 = \dfrac{6!}{3!3!} = \dfrac{6 \cdot 5 \cdot 4}{6} = 20$

49. ${}_4C_1 = \dfrac{4!}{1!3!} = 4$

50. ${}_5C_1 = \dfrac{5!}{1!4!} = 5$

51. ${}_{25}C_0 = \dfrac{25!}{0!25!} = 1$

52. ${}_{20}C_0 = \dfrac{20!}{0!20!} = 1$

53. ${}_{20}C_4 = 4845$

54. ${}_{10}C_7 = 120$

55. ${}_{42}C_5 = 850{,}668$

56. ${}_{50}C_6 = 15{,}890{,}700$

57. AB, AC, AD, AE, AF,
BC, BD, BE, BF, CD
CE, CF, DE, DF, EF

${}_6C_2 = 15$ ways

58. ABC, ABD, ABE, ABF, ACD, ACE, ACF
ADE, ADF, AEF, BCD, BCE, BCF, BDE
BDF, BEF, CDE, CDF, CEF, DEF

${}_6C_3 = 20$ ways

59. ${}_{100}C_{14} = \dfrac{100!}{14!86!} \approx 4.42 \times 10^{16}$ ways

60. ${}_{14}C_{12} = 91$ ways

61. ${}_{49}C_6 = 13{,}983{,}816$ ways

62. $({}_{55}C_5)({}_{42}C_1) = (3{,}478{,}761)(42)$

$= 146{,}107{,}962$ combinations

63. ${}_9C_2 = 36$ lines

64. There are 22 good sets and 3 defective sets.

(a) ${}_{22}C_4 = 7315$ ways

(b) $({}_{22}C_2)({}_3C_2) = (231)(3) = 693$ ways

(c) ${}_{22}C_4 + ({}_{22}C_3)({}_3C_1) + ({}_{22}C_2)({}_3C_2) = 7315 + (1540)(3) + 693 = 12{,}628$ ways

65. Select type of card for three of a kind: ${}_{13}C_1$

Select three of four cards for three of a kind: ${}_4C_3$

Select type of card for pair: ${}_{12}C_1$

Select two of four cards for pair: ${}_4C_2$

${}_{13}C_1 \cdot {}_4C_3 \cdot {}_{12}C_1 \cdot {}_4C_2 = 13 \cdot 4 \cdot 12 \cdot 6 = 3744$ ways to get a full house

66. Select 2 jacks: ${}_4C_2 = 6$

Select 3 aces: ${}_4C_3 = 4$

Total: $6 \cdot 4 = 24$ ways

67. (a) ${}_{12}C_4 = 495$ ways

(b) $({}_5C_2)({}_7C_2) = (10)(21) = 210$ ways

68. $({}_{13}C_7)({}_{20}C_3) = 1716 \cdot 1140 = 1{,}956{,}240$ ways

69. $({}_7C_1)({}_{12}C_3)({}_{20}C_2) = 7 \cdot 220 \cdot 190$
$= 292{,}600$ ways

70. (a) ${}_3C_2 = \dfrac{3!}{2!1!} = 3$ relationships

(b) ${}_8C_2 = \dfrac{8!}{2!6!} = \dfrac{8 \cdot 7}{2} = 28$ relationships

(c) ${}_{12}C_2 = \dfrac{12!}{2!10!} = \dfrac{12 \cdot 11}{2} = 66$ relationships

(d) ${}_{20}C_2 = \dfrac{20!}{2!18!} = \dfrac{20 \cdot 19}{2} = 190$ relationships

71. ${}_5C_2 - 5 = 10 - 5 = 5$ diagonals

72. ${}_6C_2 - 6 = 15 - 6 = 9$ diagonals

73. ${}_8C_2 - 8 = 28 - 8 = 20$ diagonals

74. ${}_{10}C_2 - 10 = 45 - 10 = 35$ diagonals

75. $14 \cdot {}_nP_3 = {}_{n+2}P_4$

Note: $n \geq 3$ for this to be defined.

$$14\left[\frac{n!}{(n-3)!}\right] = \frac{(n+2)!}{(n-2)!}$$

$$14n(n-1)(n-2) = (n+2)(n+1)n(n-1) \quad \text{(We can divide here by } n(n-1) \text{ since } n \neq 0, n \neq 1.)$$

$$14n - 28 = n^2 + 3n + 2$$

$$0 = n^2 - 11n + 30$$

$$0 = (n-5)(n-6)$$

$$n = 5 \quad \text{or} \quad n = 6$$

76. ${}_nP_5 = 18 \cdot {}_{n-2}P_4$

Note: $n \geq 6$ for this to be defined.

$$\frac{n!}{(n-5)!} = 18\left(\frac{(n-2)!}{(n-6)!}\right) \qquad \left(\begin{array}{l}\text{We can divide by } (n-2), (n-3),\\ (n-4) \text{ since } n \neq 2, n \neq 3, \text{ and } n \neq 4.\end{array}\right)$$

$$n(n-1)(n-2)(n-3)(n-4) = 18(n-2)(n-3)(n-4)(n-5)$$

$$n^2 - n = 18n - 90$$

$$n^2 - 19n + 90 = 0$$

$$(n-9)(n-10) = 0$$

$$n = 9 \text{ or } n = 10$$

77. ${}_nP_4 = 10 \cdot {}_{n-1}P_3$

$$\frac{n!}{(n-4)!} = 10\frac{(n-1)!}{(n-4)!}$$

$$n! = 10(n-1)!$$

$$n = 10$$

78. ${}_nP_6 = 12 \cdot {}_{n-1}P_5$

$$\frac{n!}{(n-6)!} = 12\frac{(n-1)!}{(n-6)!}$$

$$n! = 12(n-1)!$$

$$n = 12$$

79. ${}_{n+1}P_3 = 4 \cdot {}_nP_2$

$$\frac{(n+1)!}{(n-2)!} = 4\frac{n!}{(n-2)!}$$

$$(n+1)! = 4n!$$

$$n = 3$$

80. $_{n+2}P_3 = 6 \cdot {}_{n+2}P_1$

$$\frac{(n+2)!}{(n-1)!} = 6\frac{(n+2)!}{(n+1)!}$$

$$(n+1)! = 6(n-1)!$$

$$(n+1)(n) = 6$$

$$n = 2$$

81. $4 \cdot {}_{n+1}P_2 = {}_{n+2}P_3$

$$4\frac{(n+1)!}{(n-1)!} = \frac{(n+2)!}{(n-1)!}$$

$$4(n+1)! = (n+2)!$$

$$n = 2$$

82. $5 \cdot {}_{n-1}P_1 = {}_nP_2$

$$5\frac{(n-1)!}{(n-2)!} = \frac{n!}{(n-2)!}$$

$$5(n-1)! = n!$$

$$n = 5$$

83. False

84. True

85. $_{100}P_{80} \approx 3.836 \times 10^{139}$.

This number is too large for some calculators to evaluate.

86. The symbol $_nP_r$ means the number of ways to choose and order r elements out of a set of n elements.

87. $_nC_r = {}_nC_{n-r} = \dfrac{n!}{r!(n-r)!}$

88. (b) $_{10}P_6$ is larger than $_{10}C_6$ because the permutations count different orderings as distinct.

89. $_nP_{n-1} = \dfrac{n!}{(n-(n-1))!} = \dfrac{n!}{1!} = \dfrac{n!}{0!} = {}_nP_n$

90. $_nC_n = \dfrac{n!}{(n-n)!n!}$

$$= \frac{n!}{0!n!}$$

$$= \frac{n!}{n!0!} = \frac{n!}{(n-0)!0!} = {}_nC_0$$

91. $_nC_{n-1} = \dfrac{n!}{[n-(n-1)]!(n-1)!}$

$$= \frac{n!}{(1)!(n-1)!}$$

$$= \frac{n!}{(n-1)!1!} = {}_nC_1$$

92. $_nC_r = \dfrac{n!}{(n-r)!r!}$

$$= \frac{1}{r!}\left[\frac{n!}{(n-r)!}\right]$$

$$= \frac{{}_nP_r}{r!}$$

93. From the graph of $y = \sqrt{x-3} - x + 6$, you see that there is one zero, $x \approx 8.303$. Analytically,

$$\sqrt{x-3} = x - 6$$

$$x - 3 = x^2 - 12x + 36$$

$$0 = x^2 - 13x + 39.$$

By the Quadratic Formula, $x = \dfrac{13 \pm \sqrt{(-13)^2 - 4(39)}}{2} = \dfrac{13 \pm \sqrt{13}}{2}$.

Selecting the larger solution, $x = \dfrac{13 + \sqrt{13}}{2} \approx 8.303$. (The other solution is extraneous.)

94. $\dfrac{4}{t} + \dfrac{3}{2t} = 1$

$$\frac{8+3}{2t} = 1$$

$$11 = 2t$$

$$t = \frac{11}{2} = 5.5$$

95. $\log_2(x-3) = 5$

$$2^5 = x - 3$$

$$2^5 + 3 = x$$

$$x = 35$$

96. $e^{x/3} = 16$

$$\frac{x}{3} = \ln 16$$

$$x = 3 \ln 16 \approx 8.318$$

97. $x = \dfrac{\begin{vmatrix} -14 & 3 \\ 2 & -2 \end{vmatrix}}{\begin{vmatrix} -5 & 3 \\ 7 & -2 \end{vmatrix}} = \dfrac{22}{-11} = -2$

$y = \dfrac{\begin{vmatrix} -5 & -14 \\ 7 & 2 \end{vmatrix}}{\begin{vmatrix} -5 & 3 \\ 7 & -2 \end{vmatrix}} = \dfrac{88}{-11} = -8$

Answer: $(-2, -8)$

98. $x = \dfrac{\begin{vmatrix} 35 & 1 \\ 10 & 2 \end{vmatrix}}{\begin{vmatrix} 8 & 1 \\ 6 & 2 \end{vmatrix}} = \dfrac{60}{10} = 6$

$y = \dfrac{\begin{vmatrix} 8 & 35 \\ 6 & 10 \end{vmatrix}}{\begin{vmatrix} 8 & 1 \\ 6 & 2 \end{vmatrix}} = \dfrac{-130}{10} = -13$

Answer: $(6, -13)$

99. $x = \dfrac{\begin{vmatrix} -1 & -4 \\ -4 & 5 \end{vmatrix}}{\begin{vmatrix} -3 & -4 \\ 9 & 5 \end{vmatrix}} = \dfrac{-21}{21} = -1$

$y = \dfrac{\begin{vmatrix} -3 & -1 \\ 9 & -4 \end{vmatrix}}{\begin{vmatrix} -3 & -4 \\ 9 & 5 \end{vmatrix}} = \dfrac{21}{21} = 1$

Answer: $(-1, 1)$

100. $x = \dfrac{\begin{vmatrix} -74 & -11 \\ 8 & -4 \end{vmatrix}}{\begin{vmatrix} 10 & -11 \\ -8 & -4 \end{vmatrix}} = \dfrac{384}{-128} = -3$

$y = \dfrac{\begin{vmatrix} 10 & -74 \\ -8 & 8 \end{vmatrix}}{\begin{vmatrix} 10 & -11 \\ -8 & -4 \end{vmatrix}} = \dfrac{-512}{-128} = 4$

Answer: $(-3, 4)$

Section 8.7 Probability

You should know the following basic principles of probability.

- If an event E has $n(E)$ equally likely outcomes and its sample space has $n(S)$ equally likely outcomes, then the probability of event E is

  $P(E) = \dfrac{n(E)}{n(S)}$, where $0 \le P(E) \le 1$.

- If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

  If A and B are not mutually exclusive events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- If A and B are independent events, then the probability that both A and B will occur is $P(A)P(B)$.
- The probability of the complement of an event A is $P(A') = 1 - P(A)$.

Vocabulary Check

1. experiment, outcomes
2. sample space
3. probability
4. impossible, certain
5. mutually exclusive
6. independent
7. complement
8. (a) iii (b) i (c) iv (d) ii

1. $\{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

2. $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

3. $\{ABC, ACB, BAC, BCA, CAB, CBA\}$

4. $\{(R, R), (R, B), (R, Y), (B, B), (B, Y), (B, R), (Y, B), (Y, R)\}$

5. $\{(A, B), (A, C), (A, D), (A, E), (B, C), (B, D), (B, E), (C, D), (C, E), (D, E)\}$

6. $\{SSS, SSF, SFS, FSS, SFF, FFS, FSF, FFF\}$

7. $E = \{HTT, THT, TTH\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$$

8. $E = \{HHH, HHT, HTH, HTT\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

9. $E = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{7}{8}$$

10. $E = \{HHH, HHT, HTH, THH\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

11. $E = \{K, K, K, K, Q, Q, Q, Q, J, J, J, J\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{12}{52} = \frac{3}{13}$$

12. The probability that the card is not a black face card is the complement of getting a black face card.

$E = \{K, K, Q, Q, J, J\}$

Hence, $P(E) = \frac{6}{52}$ and

$P(E') = 1 - P(E) = 1 - \frac{6}{52} = \frac{23}{26}$.

13. $E = \{A, A, A, A, K, K, K, K, Q, Q, Q, Q, J, J, J, J\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{16}{52} = \frac{4}{13}$$

14. There are 9 possible cards in each of 4 suits.

$9 \cdot 4 = 36$

$$P(E) = \frac{n(E)}{n(S)} = \frac{36}{52} = \frac{9}{13}$$

15. $E = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{5}{36}$$

16. $E = \{(2, 6), (3, 5), (3, 6), (4, 4), (4, 5), (4, 6), (5, 3), (5, 4), (5, 5), (5, 6), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{15}{36} = \frac{5}{12}$$

17. not $E = \{(5, 6), (6, 5), (6, 6)\}$

$n(E) = n(S) - n(\text{not } E) = 36 - 3 = 33$

$$P(E) = \frac{n(E)}{n(S)} = \frac{33}{36} = \frac{11}{12}$$

18. $E = \{(1, 1), (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (3, 6), (4, 1), (4, 3), (4, 5), (5, 2), (5, 4), (5, 6), (6, 1), (6, 3), (6, 5)\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{19}{36}$$

19. $P(E) = \dfrac{{}_3C_2}{{}_6C_2} = \dfrac{3}{15} = \dfrac{1}{5}$

20. $P(E) = \dfrac{{}_2C_2}{{}_6C_2} = \dfrac{1}{15}$

21. $P(E) = \dfrac{{}_4C_2}{{}_6C_2} = \dfrac{6}{15} = \dfrac{2}{5}$

22. $$P(E) = \frac{{}_1C_1 \cdot {}_2C_1 + {}_1C_1 \cdot {}_3C_1 + {}_2C_1 \cdot {}_3C_1}{{}_6C_2} = \frac{2 + 3 + 6}{15} = \frac{11}{15}$$

23. $P(E') = 1 - P(E) = 1 - 0.75 = 0.25$

24. $P(E') = 1 - P(E) = 1 - 0.\overline{2} = 1 - \frac{2}{9} = \frac{7}{9} = 0.\overline{7}$

25. $P(E') = 1 - P(E) = 1 - \frac{2}{3} = \frac{1}{3}$

26. $P(E') = 1 - P(E) = 1 - \frac{7}{8} = \frac{1}{8}$

27. $P(E) = 1 - P(E') = 1 - p = 1 - 0.12 = 0.88$

28. $P(E) = 1 - P(E') = 1 - p = 1 - 0.84 = 0.16$

29. $P(E) = 1 - P(E') = 1 - \frac{13}{20} = \frac{7}{20}$

30. $P(E) = 1 - P(E') = 1 - \frac{61}{100} = \frac{39}{100}$

31. (a) $0.15(8.15) \approx 1.22$ million

(b) $\dfrac{0.41}{1.0} = 0.41$

(c) $\dfrac{0.24}{1.0} = 0.24$

(d) $\dfrac{0.24 + 0.02}{1.0} = 0.26$

32. (a) $0.10(42) \approx 4$ presidents had no children.

(b) $0.19(42) \approx 8$ presidents had four children.

(c) $(0.07 + 0.21) = 0.28$

(d) 0.14

Answers will vary.

33. (a) $(0.128)(293.66) \approx 37.6$ million

(b) $\dfrac{0.01}{1.0} = 0.01$

(c) $\dfrac{0.01 + 0.002}{1.0} = 0.012$

34. (a) $1 - 0.148 = 0.852$ probability of having a high school diploma. Hence,

$0.852(186.88) \approx 159.2$ million.

(b) $0.097(186.88) \approx 18.1$ million

(c) $0.181 + 0.097 = 0.278$

(d) $1 - 0.148 = 0.852$

(e) $0.084 + 0.181 + 0.097 = 0.362$

35. (a) $\dfrac{34}{100} = 0.34$

(b) $\dfrac{45}{100} = 0.45$

(c) $\dfrac{23}{100} = 0.23$

36. (a) $\dfrac{290}{500} = 0.58$

(b) $\dfrac{478}{500} = 0.956$

(c) $\dfrac{2}{500} = 0.004$

37. (a) $\dfrac{672}{1254}$

(b) $\dfrac{582}{1254}$

(c) $\dfrac{672 - 124}{1254} = \dfrac{548}{1254}$

38. (a) $\dfrac{48 + 56}{128} = \dfrac{104}{128} = \dfrac{13}{16}$

(b) $\dfrac{4 + 20}{128} = \dfrac{24}{128} = \dfrac{3}{16}$ $\left[\textbf{Note: } 1 - \dfrac{13}{16} = \dfrac{3}{16}\right]$

(c) $\dfrac{4}{128} = \dfrac{1}{32}$

39. $p + p + 2p = 1$

$p = 0.25$

Taylor: $0.50 = \dfrac{1}{2}$, Moore: $0.25 = \dfrac{1}{4}$

Perez: $0.25 = \dfrac{1}{4}$

40. $\dfrac{54}{31 + 54 + 42 + 20 + 47 + 58} = \dfrac{54}{252} = \dfrac{3}{14}$

41. (a) $\dfrac{{}_{15}C_{10}}{{}_{20}C_{10}} = \dfrac{3003}{184,756} = \dfrac{21}{1292} \approx 0.016$

(b) $\dfrac{{}_{15}C_8 \cdot {}_5C_2}{{}_{20}C_{10}} = \dfrac{64,350}{184,756} = \dfrac{225}{646} \approx 0.348$

(c) $\dfrac{{}_{15}C_9 \cdot {}_5C_1}{{}_{20}C_{10}} + \dfrac{{}_{15}C_{10}}{{}_{20}C_{10}} = \dfrac{25,025 + 3003}{184,756}$

$= \dfrac{28,028}{184,756} = \dfrac{49}{323} \approx 0.152$

42. Total ways to insert paychecks: $5! = 120$ ways

5 correct: 1 way

4 correct: not possible

3 correct: 10 ways

2 correct: 20 ways

1 correct: 45 ways

0 correct: 44 ways

(a) $\dfrac{45}{120} = \dfrac{3}{8}$ (b) $\dfrac{45 + 20 + 10 + 1}{120} = \dfrac{19}{30}$

43. (a) $\dfrac{1}{_5P_5} = \dfrac{1}{120}$

(b) $\dfrac{1}{_4P_4} = \dfrac{1}{24}$

44. (a) $\dfrac{(_8C_2)(_{100}C_5)}{_{108}C_7} = 0.0756$

(b) $\dfrac{(_8C_2)(_{25}C_2)(_{25}C_3)}{_{108}C_7} \approx 6.929 \times 10^{-4}$

45. (a) There are three letters to be selected, and two must be Q and Y.

QY__, YQ__, Q__Y, Y__Q, __YQ, __QY

Thus, the probability is

$\dfrac{6(26)}{26^3} = \dfrac{6}{26^2} \approx 0.008876.$

(b) The three letters must be Q, Y, and X.

QYX, QXY, YQX, YXQ, XQY, XYQ

Thus, the probability is $\dfrac{6}{26^3} = \dfrac{3}{8788}$.

46. (a) $\dfrac{1}{10^4} = 0.0001$

(b) $\dfrac{1}{10^2} = 0.01$

47. (a) $\dfrac{100}{(_{55}C_5)(_{42}C_1)} = \dfrac{100}{(3{,}478{,}761)(42)}$

(b) $\dfrac{1000}{(_{55}C_5)(_{42}C_1)} = \dfrac{1000}{(3{,}478{,}761)(42)}$

48. (a) $\dfrac{1}{10^9}$

(b) $\dfrac{1}{10^4}$

(c) $\dfrac{1}{10^2}$

49. (a) $\dfrac{20}{52} = \dfrac{5}{13}$

(b) $\dfrac{13 + 13}{52} = \dfrac{1}{2}$

(c) $\dfrac{4 + 12}{52} = \dfrac{4}{13}$

50. $\dfrac{_{13}C_1 \cdot {_4C_3} \cdot {_{12}C_1} \cdot {_4C_2}}{_{52}C_5} = \dfrac{13 \cdot 4 \cdot 12 \cdot 6}{2{,}598{,}960}$

$= \dfrac{3744}{2{,}598{,}960}$

$= \dfrac{6}{4165}$

51. (a) $\dfrac{_9C_4}{_{12}C_4} = \dfrac{126}{495} = \dfrac{14}{55}$ (4 good units)

(b) $\dfrac{(_9C_2)(_3C_2)}{_{12}C_4} = \dfrac{108}{495} = \dfrac{12}{55}$ (2 good units)

(c) $\dfrac{(_9C_3)(_3C_1)}{_{12}C_4} = \dfrac{252}{495} = \dfrac{28}{55}$ (3 good units)

At least 2 good units: $\dfrac{12}{55} + \dfrac{28}{55} + \dfrac{14}{55} = \dfrac{54}{55}$

52. (a) $P(EE) = \dfrac{20}{40} \cdot \dfrac{20}{40} = \dfrac{1}{4}$

(b) $P(EO \text{ or } OE) = 2\left(\dfrac{20}{40}\right)\left(\dfrac{20}{40}\right) = \dfrac{1}{2}$

(c) $P(N_1 < 30, N_2 < 30) = \dfrac{29}{40} \cdot \dfrac{29}{40} = \dfrac{841}{1600}$

(d) $P(N_1N_1) = \dfrac{40}{40} \cdot \dfrac{1}{40} = \dfrac{1}{40}$

53. $(0.32)^2 = 0.1024$

54. $(0.78)^3 = 0.474552$

55. (a) $P(SS) = (0.985)^2 \approx 0.9702$

(b) $P(S) = 1 - P(FF) = 1 - (0.015)^2 \approx 0.9998$

(c) $P(FF) = (0.015)^2 \approx 0.0002$

56. (a) $P(AA) = (0.90)^2 = 0.81$

(b) $P(NN) = (0.10)^2 = 0.01$

(c) $P(A) = 1 - P(NN) = 1 - 0.01 = 0.99$

57. (a) $\left(\dfrac{1}{5}\right)^6 = \dfrac{1}{15{,}625}$

(b) $\left(\dfrac{4}{5}\right)^6 = \dfrac{4096}{15{,}625} = 0.262144$

(c) $1 - 0.262144 = 0.737856 = \dfrac{11{,}529}{15{,}625}$

58. (a) $P(BBBB) = \left(\dfrac{1}{2}\right)^4 = \dfrac{1}{16}$

(b) $P(BBBB) + P(GGGG) = \left(\dfrac{1}{2}\right)^4 + \left(\dfrac{1}{2}\right)^4 = \dfrac{1}{8}$

(c) $$\begin{aligned} P(\text{at least one boy}) &= 1 - P(\text{no boys}) \\ &= 1 - P(GGGG) \\ &= 1 - \frac{1}{16} = \frac{15}{16} \end{aligned}$$

59. (a) If the *center* of the coin falls within the circle of radius $d/2$ around a vertex, the coin will cover the vertex.

$$P(\text{coin covers a vertex}) = \frac{\text{Area in which coin may fall so that it covers a vertex}}{\text{Total area}} = \frac{n\left[\pi\left(\frac{d}{2}\right)^2\right]}{nd^2} = \frac{1}{4}\pi$$

(b) Experimental results will vary.

60. $1 - \dfrac{(45)^2}{(60)^2} = 1 - \left(\dfrac{45}{60}\right)^2 = 1 - \left(\dfrac{3}{4}\right)^2 = 1 - \dfrac{9}{16} = \dfrac{7}{16}$

61. True

$P(E) + P(E') = 1$

62. False. The first sentence is true, but the second is false. The complement is to roll a number greater than 2, and its probability is $\frac{2}{3}$.

63. (a) As you consider successive people with distinct birthdays, the probabilities must decrease to take into account the birth dates already used. Since the birth dates of people are independent events, multiply the respective probabilities of distinct birthdays.

(b) $\dfrac{365}{365} \cdot \dfrac{364}{365} \cdot \dfrac{363}{365} \cdot \dfrac{362}{365}$

(c) $P_1 = \dfrac{365}{365} = 1$

$$P_2 = \frac{365}{365} \cdot \frac{364}{365} = \frac{364}{365} P_1 = \frac{365 - (2 - 1)}{365} P_1$$

$$P_3 = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} = \frac{363}{365} P_2 = \frac{365 - (3 - 1)}{365} P_2$$

$$P_n = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \cdots \cdot \frac{365 - (n - 1)}{365} = \frac{365 - (n - 1)}{365} P_{n-1}$$

(d) Q_n is the probability that the birthdays are *not* distinct which is equivalent to at least 2 people having the same birthday.

—CONTINUED—

63. —CONTINUED—

(e)

n	10	15	20	23	30	40	50
P_n	0.88	0.75	0.59	0.49	0.29	0.11	0.03
Q_n	0.12	0.25	0.41	0.51	0.71	0.89	0.97

(f) 23, See the chart above.

64. If a weather forecast indicates that the probability of rain is 40%, this means the meteorological records indicate that over an extended period of time with similar weather conditions it will rain 40% of the time.

65. $\dfrac{2}{x-5} = 4$

$2 = 4(x-5) = 4x - 20$

$4x = 22$

$x = \dfrac{11}{2}$

66. $\dfrac{3}{2x+3} - 4 = \dfrac{-1}{2x+3}$

$\dfrac{4}{2x+3} = 4$

$1 = 2x + 3$

$2x = -2$

$x = -1$

67. $\dfrac{3}{x-2} + \dfrac{x}{x+2} = 1$

$3(x+2) + x(x-2) = (x-2)(x+2)$

$3x + 6 + x^2 - 2x = x^2 - 4$

$x = -10$

68. $\dfrac{2}{x} - \dfrac{5}{x-2} = \dfrac{-13}{x^2-2x} = \dfrac{-13}{x(x-2)}$

$2(x-2) - 5(x) = -13$

$-3x = -9$

$x = 3$

69. $e^x + 7 = 35$

$e^x = 28$

$x = \ln(28) \approx 3.332$

70. $200e^{-x} = 75$

$e^{-x} = \frac{75}{200} = \frac{3}{8}$

$-x = \ln\left(\frac{3}{8}\right)$

$x = -\ln\left(\frac{3}{8}\right) = \ln\left(\frac{8}{3}\right) \approx 0.981$

71. $4 \ln 6x = 16$

$\ln 6x = 4$

$e^4 = 6x$

$x = \frac{1}{6}e^4 \approx 9.10$

72. $5 \ln 2x - 4 = 11$

$\ln 2x = 3$

$2x = e^3$

$x = \frac{1}{2}e^3 \approx 10.043$

73. ${}_5P_3 = \dfrac{5!}{(5-3)!} = \dfrac{120}{2} = 60$

74. ${}_{10}P_4 = \dfrac{10!}{(10-4)!} = \dfrac{10!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$

75. ${}_{11}P_8 = \dfrac{11!}{(11-8)!} = \dfrac{11!}{3!} = 6{,}652{,}800$

76. ${}_9P_2 = \dfrac{9!}{(9-2)!} = \dfrac{9!}{7!} = 9 \cdot 8 = 72$

77. ${}_6C_2 = \dfrac{6!}{4!2!} = \dfrac{6 \cdot 5 \cdot 4!}{4!2} = 15$

78. ${}_9C_5 = 126$

79. $_{11}C_8 = \frac{11!}{8!3!} = \frac{11 \cdot 10 \cdot 9 \cdot 8!}{8!6} = 165$

80. $_{16}C_{13} = 560$

Review Exercises for Chapter 8

1. $a_n = \frac{2^n}{2^n + 1}$

$a_1 = \frac{2^1}{2^1 + 1} = \frac{2}{3}$

$a_2 = \frac{2^2}{2^2 + 1} = \frac{4}{5}$

$a_3 = \frac{2^3}{2^3 + 1} = \frac{8}{9}$

$a_4 = \frac{2^4}{2^4 + 1} = \frac{16}{17}$

$a_5 = \frac{2^5}{2^5 + 1} = \frac{32}{33}$

2. $a_n = \frac{1}{n} - \frac{1}{n+1}$

$a_1 = \frac{1}{1} - \frac{1}{2} = \frac{1}{2}$

$a_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$a_3 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$a_4 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$

$a_5 = \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$

3. $a_n = \frac{(-1)^n}{n!}$

$a_1 = \frac{(-1)^1}{1!} = -1$

$a_2 = \frac{(-1)^2}{2!} = \frac{1}{2}$

$a_3 = \frac{(-1)^3}{3!} = -\frac{1}{6}$

$a_4 = \frac{(-1)^4}{4!} = \frac{1}{24}$

$a_5 = \frac{(-1)^5}{5!} = -\frac{1}{120}$

4. $a_n = \frac{(-1)^n}{(2n+1)!}$

$a_1 = \frac{(-1)^1}{3!} = -\frac{1}{6}$

$a_2 = \frac{(-1)^2}{5!} = \frac{1}{120}$

$a_3 = \frac{(-1)^3}{7!} = \frac{-1}{5040}$

$a_4 = \frac{(-1)^4}{9!} = \frac{1}{362,880}$

$a_5 = \frac{(-1)^5}{11!} = \frac{-1}{39,916,800}$

5. Common difference is 5.

$a_n = 5n, n = 1, 2, \ldots$

6. Common difference is -2.

$a_n = 52 - 2n, n = 1, 2, 3, \ldots$

7. Denominators are successive odd numbers.

$a_n = \frac{2}{2n - 1}, n = 1, 2, 3, \ldots$

8. $a_n = \frac{n+2}{n+1}, n = 1, 2, 3, \ldots$

9. $a_1 = 9, a_{k+1} = a_k - 4$

$a_2 = a_1 - 4 = 9 - 4 = 5$

$a_3 = 5 - 4 = 1$

$a_4 = 1 - 4 = -3$

$a_5 = -3 - 4 = -7$

10. $a_1 = 49, a_{k+1} = a_k + 6$

$a_2 = a_1 + 6 = 49 + 6 = 55$

$a_3 = 55 + 6 = 61$

$a_4 = 67$

$a_5 = 73$

11. $\frac{18!}{20!} = \frac{18!}{20 \cdot 19 \cdot 18!}$

$= \frac{1}{20 \cdot 19} = \frac{1}{380}$

12. $\frac{10!}{8!} = \frac{10 \cdot 9 \cdot 8!}{8!} = 90$

13. $\dfrac{(n+1)!}{(n-1)!} = \dfrac{(n+1)n(n-1)!}{(n-1)!} = n(n+1)$

14. $\dfrac{2n!}{(n+1)!} = \dfrac{2n!}{(n+1)n!} = \dfrac{2}{n+1}$

15. $\displaystyle\sum_{i=1}^{6} 5 = 6(5) = 30$

16. $\displaystyle\sum_{k=2}^{5} 4k = 8 + 12 + 16 + 20 = 56$

17. $\displaystyle\sum_{j=1}^{4} \frac{6}{j^2} = \frac{6}{1^2} + \frac{6}{2^2} + \frac{6}{3^2} + \frac{6}{4^2}$

$= 6 + \dfrac{3}{2} + \dfrac{2}{3} + \dfrac{3}{8} = \dfrac{205}{24}$

18. $\displaystyle\sum_{i=1}^{8} \frac{i}{i+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \frac{7}{8} + \frac{8}{9}$

≈ 6.17

19. $\displaystyle\sum_{k=1}^{100} 2k^3 = 2 \cdot \frac{100^2(101)^2}{4} = 51{,}005{,}000$

20. $\displaystyle\sum_{j=0}^{40} (j^2 + 1) = \frac{40(41)(81)}{6} + 41 = 22{,}181$

21. $\displaystyle\sum_{n=0}^{50} (n^2 + 3) = \frac{50(51)(101)}{6} + 3(51) = 43{,}078$

22. $\displaystyle\sum_{n=1}^{100} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right) + \left(\frac{1}{100} - \frac{1}{101}\right)$

$= \dfrac{1}{1} - \dfrac{1}{101} = \dfrac{100}{101}$

23. $\displaystyle\frac{1}{2(1)} + \frac{1}{2(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{2(20)} = \sum_{k=1}^{20} \frac{1}{2k}$

≈ 1.799

24. $\displaystyle 2(1^2) + 2(2^2) + 2(3^2) + \cdots + 2(9^2) = \sum_{k=1}^{9} 2k^2$

$= 570$

25. $\displaystyle\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{9}{10} = \sum_{k=1}^{9} \frac{k}{k+1} \approx 7.071$

26. $\displaystyle 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots = \sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k = \frac{3}{4}$

27. (a) $\displaystyle\sum_{k=1}^{4} \frac{5}{10^k} = \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \frac{5}{10{,}000} = 0.5 + 0.05 + 0.005 + 0.0005 = 0.5555 = \frac{1111}{2000}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{5}{10^k} = \frac{5}{10} \sum_{k=0}^{\infty} \frac{1}{10^k} = \frac{5}{10} \cdot \frac{1}{1 - 1/10} = \frac{5}{10} \cdot \frac{10}{9} = \frac{5}{9}$

28. $\displaystyle\sum_{k=1}^{\infty} \frac{3}{2^k}$

(a) $\displaystyle\sum_{k=1}^{4} \frac{3}{2^k} = \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} = \frac{45}{16} = 2.8125$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{2^k} = \sum_{k=0}^{\infty} \left(\frac{3}{2}\right)\left(\frac{1}{2^k}\right) = \frac{3/2}{1 - 1/2} = 3$

29. $\displaystyle\sum_{k=1}^{\infty} 2(0.5)^k$

(a) $\displaystyle\sum_{k=1}^{4} 2(0.5)^k = 2(0.5) + 2(0.5)^2 + 2(0.5)^3 + 2(0.5)^4$

$= 1.875 = \dfrac{15}{8}$

(b) $\displaystyle\sum_{k=1}^{\infty} 2(0.5)^k = 2(0.5)\frac{1}{1 - 0.5} = 2$

30. $\sum_{k=1}^{\infty} 4(0.25)^k$

(a) $\sum_{k=1}^{\infty} 4(0.25)^k = 4(0.25) + 4(0.25)^2 + 4(0.25)^3 + 4(0.25)^4 = 1.328125 = \frac{85}{64}$

(b) $\sum_{k=1}^{\infty} 4(0.25)^k = \sum_{k=0}^{\infty} 4(0.25)(0.25)^k = \frac{1}{1 - 0.25} = \frac{4}{3}$

31. $a_n = 2500\left(1 + \frac{0.02}{4}\right)^n, n = 1, 2, 3$

(a) $a_1 = 2500\left(1 + \frac{0.02}{4}\right)^1 = 2512.5$

$a_2 = 2525.06 \quad a_3 = 2537.69$

$a_4 = 2550.38 \quad a_5 = 2563.13$

$a_6 = 2575.94 \quad a_7 = 2588.82$

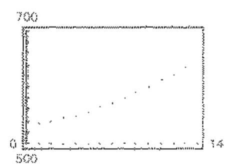
$a_8 = 2601.77$

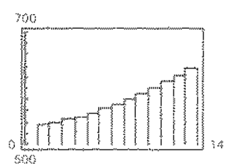
(b) $a_{40} = 2500\left(1 + \frac{0.02}{4}\right)^{40} = \$3051.99$

32. (a)

n	1	2	3	4	5	6
a_n	535	539	544	549	556	563

n	7	8	9	10	11	12	13
a_n	571	580	590	600	611	623	636

(b)


(c)


(d) For 2004, $n = 14$ and $a_{14} \approx 650$ thousand.

For 2010, $n = 20$ and $a_{20} \approx 750$ thousand.

The results seem reasonable.

33. Yes

$d = 3 - 5 = -2$

34. Not arithmetic

35. Yes

$d = 1 - \frac{1}{2} = \frac{1}{2}$

36. Arithmetic

$d = \frac{8}{9} - \frac{9}{9} = \frac{-1}{9}$

37. $a_1 = 3, d = 4$

$a_1 = 3$

$a_2 = 3 + 4 = 7$

$a_3 = 7 + 4 = 11$

$a_4 = 11 + 4 = 15$

$a_5 = 15 + 4 = 19$

38. $a_1 = 8, d = -2$

$a_1 = 8$

$a_2 = 8 - 2 = 6$

$a_3 = 6 - 2 = 4$

$a_4 = 4 - 2 = 2$

$a_5 = 2 - 2 = 0$

39. $a_4 = 10,\ a_{10} = 28$

$a_{10} = a_4 + 6d$

$28 = 10 + 6d$

$18 = 6d$

$3 = d$

$a_1 = a_4 - 3d$

$a_1 = 10 - 3(3)$

$a_1 = 1$

$a_2 = 1 + 3 = 4$

$a_3 = 4 + 3 = 7$

$a_4 = 7 + 3 = 10$

$a_5 = 10 + 3 = 13$

40. $a_2 = 14, a_6 = 22$

$a_6 = a_2 + 4d$

$22 = 14 + 4d$

$8 = 4d$

$2 = d$

$a_1 = a_2 - d$

$a_1 = 14 - 2 = 12$

$a_2 = 12 + 2 = 14$

$a_3 = 14 + 2 = 16$

$a_4 = 16 + 2 = 18$

$a_5 = 18 + 2 = 20$

41. $a_1 = 35, a_{k+1} = a_k - 3$

$a_1 = 35$

$a_2 = a_1 - 3 = 35 - 3 = 32$

$a_3 = a_2 - 3 = 32 - 3 = 29$

$a_4 = a_3 - 3 = 29 - 3 = 26$

$a_5 = a_4 - 3 = 26 - 3 = 23$

$a_n = 35 + (n - 1)(-3) = 38 - 3n, d = -3$

42. $a_1 = 15, a_{k+1} = a_k + \frac{5}{2}$

$a_1 = 15$

$a_2 = 15 + \frac{5}{2} = \frac{35}{2}$

$a_3 = \frac{35}{2} + \frac{5}{2} = \frac{40}{2} = 20$

$a_4 = 20 + \frac{5}{2} = \frac{45}{2}$

$a_5 = \frac{45}{2} + \frac{5}{2} = \frac{50}{2} = 25$

$a_n = 15 + \frac{5}{2}(n - 1) = \frac{25}{2} + \frac{5}{2}n, d = \frac{5}{2}$

43. $a_1 = 9, a_{k+1} = a_k + 7$

$a_1 = 9$

$a_2 = a_1 + 7 = 9 + 7 = 16$

$a_3 = a_2 + 7 = 16 + 7 = 23$

$a_4 = a_3 + 7 = 23 + 7 = 30$

$a_5 = a_4 + 7 = 30 + 7 = 37$

$a_n = 9 + (n - 1)(7) = 2 + 7n, d = 7$

44. $a_1 = 100, a_{k+1} = a_k - 5$

$a_1 = 100$

$a_2 = 100 - 5 = 95$

$a_3 = 95 - 5 = 90$

$a_4 = 90 - 5 = 85$

$a_5 = 85 - 5 = 80$

$a_n = 100 - 5(n - 1) = 105 - 5n, d = -5$

45. $a_n = 100 + (n - 1)(-3) = 103 - 3n$

$$\sum_{n=1}^{20}(103 - 3n) = \sum_{n=1}^{20} 103 - 3\sum_{n=1}^{20} n = 20(103) - 3\left[\frac{(20)(21)}{2}\right] = 1430$$

46. $a_3 = a_1 + 2d$

$28 = 10 + 2d$

$18 = 2d$

$9 = d$

$a_n = 10 + (n - 1)9 = 1 + 9n$

$$\sum_{n=1}^{20}(1 + 9n) = \sum_{n=1}^{20} 1 + 9\sum_{n=1}^{20} n = 20(1) + 9\left[\frac{(20)(21)}{2}\right] = 1910$$

47. $$\sum_{j=1}^{10}(2j - 3) = 2\sum_{j=1}^{10} j - \sum_{j=1}^{10} 3 = 2\left[\frac{10(11)}{2}\right] - 10(3) = 80$$

48. $$\sum_{j=1}^{8}(20 - 3j) = \sum_{j=1}^{8} 20 - 3\sum_{j=1}^{8} j = 8(20) - 3\left[\frac{(8)(9)}{2}\right] = 52$$

49. $$\sum_{k=1}^{11}\left(\frac{2}{3}k + 4\right) = \frac{2}{3}\sum_{k=1}^{11} k + \sum_{k=1}^{11} 4 = \frac{2}{3} \cdot \frac{(11)(12)}{2} + 11(4) = 88$$

50. $$\sum_{k=1}^{25}\left(\frac{3k + 1}{4}\right) = \frac{3}{4}\sum_{k=1}^{25} k + \sum_{k=1}^{25} \frac{1}{4} = \frac{3}{4}\left[\frac{(25)(26)}{2}\right] + 25\left(\frac{1}{4}\right) = 250$$

51. $$\sum_{k=1}^{100} 5k = 5\left[\frac{(100)(101)}{2}\right] = 25{,}250$$

52. $$\sum_{n=20}^{80} n = \sum_{n=1}^{80} n - \sum_{n=1}^{19} n = \frac{(80)(81)}{2} - \frac{(19)(20)}{2} = 3050$$

53. (a) $34{,}000 + 4(2250) = \$43{,}000$

(b) $\sum_{k=1}^{5}[34{,}000 + (k-1)(2250)]$

$= \sum_{k=1}^{5}(31{,}750 + 2250k)$

$= \$192{,}500$

54. $a_1 = 123, d = 112 - 123 = -11$

$n = 8$

$a_8 = (-11)8 + 134 = 46$

$S_8 = \frac{8}{2}(123 + 46) = 676$ bales

55. 5, 10, 20, 40

Geometric: $r = 2$

56. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

Not geometric:

$\frac{1}{2}r = \frac{2}{3} \implies r = \frac{4}{3}$

$\frac{2}{3}r = \frac{3}{4} \implies r = \frac{9}{8}$

57. Geometric:

$r = -\frac{1}{3}$

58. Geometric:

$r = -2$

59. $a_1 = 4,\ r = -\frac{1}{4}$

$a_1 = 4$

$a_2 = 4\left(-\frac{1}{4}\right) = -1$

$a_3 = -1\left(-\frac{1}{4}\right) = \frac{1}{4}$

$a_4 = \frac{1}{4}\left(-\frac{1}{4}\right) = -\frac{1}{16}$

$a_5 = -\frac{1}{16}\left(-\frac{1}{4}\right) = \frac{1}{64}$

60. $a_1 = 2,\ r = \frac{3}{2}$

$a_1 = 2$

$a_2 = 2\left(\frac{3}{2}\right) = 3$

$a_3 = 3\left(\frac{3}{2}\right) = \frac{9}{2}$

$a_4 = \frac{9}{2}\left(\frac{3}{2}\right) = \frac{27}{4}$

$a_5 = \frac{27}{4}\left(\frac{3}{2}\right) = \frac{81}{8}$

61. $a_1 = 9,\ a_3 = 4$

$a_3 = a_1r^2$

$4 = 9r^2$

$\frac{4}{9} = r^2 \implies r = \pm\frac{2}{3}$

$a_1 = 9$

$a_2 = 9\left(\frac{2}{3}\right) = 6$

$a_3 = 6\left(\frac{2}{3}\right) = 4$

$a_4 = 4\left(\frac{2}{3}\right) = \frac{8}{3}$

$a_5 = \frac{8}{3}\left(\frac{2}{3}\right) = \frac{16}{9}$

or

$a_1 = 9$

$a_2 = 9\left(-\frac{2}{3}\right) = -6$

$a_3 = -6\left(-\frac{2}{3}\right) = 4$

$a_4 = 4\left(-\frac{2}{3}\right) = -\frac{8}{3}$

$a_5 = -\frac{8}{3}\left(-\frac{2}{3}\right) = \frac{16}{9}$

62. $a_1 = 2, a_3 = 12$

$a_3 = a_1r^2$

$12 = 2r^2$

$6 = r^2$

$\pm\sqrt{6} = r$

$a_1 = 2$

$a_2 = 2\left(\sqrt{6}\right) = 2\sqrt{6}$

$a_3 = 2\sqrt{6}\left(\sqrt{6}\right) = 12$

$a_4 = 12\left(\sqrt{6}\right) = 12\sqrt{6}$

$a_5 = 12\sqrt{6}\,\sqrt{6} = 72$

or

$a_1 = 2$

$a_2 = 2\left(-\sqrt{6}\right) = -2\sqrt{6}$

$a_3 = -2\sqrt{6}\left(-\sqrt{6}\right) = 12$

$a_4 = 12\left(-\sqrt{6}\right) = -12\sqrt{6}$

$a_5 = -12\sqrt{6}\left(-\sqrt{6}\right) = 72$

63. $a_1 = 120, a_{k+1} = \frac{1}{3}a_k$

$a_1 = 120$

$a_2 = \frac{1}{3}(120) = 40$

$a_3 = \frac{1}{3}(40) = \frac{40}{3}$

$a_4 = \frac{1}{3}\left(\frac{40}{3}\right) = \frac{40}{9}$

$a_5 = \frac{1}{3}\left(\frac{40}{9}\right) = \frac{40}{27}$

$a_n = 120\left(\frac{1}{3}\right)^{n-1},\ r = \frac{1}{3}$

64. $a_1 = 200, a_{k+1} = 0.1a_k$

$a_1 = 200$

$a_2 = 0.1(200) = 20$

$a_3 = 0.1(20) = 2$

$a_4 = 0.1(2) = 0.2$

$a_5 = 0.1(0.2) = 0.02$

$a_n = 200(0.1)^{n-1}$

65. $a_1 = 25, a_{k+1} = -\frac{3}{5}a_k$

$a_1 = 25$

$a_2 = -\frac{3}{5}(25) = -15$

$a_3 = -\frac{3}{5}(-15) = 9$

$a_4 = -\frac{3}{5}(9) = -\frac{27}{5}$

$a_5 = -\frac{3}{5}\left(-\frac{27}{5}\right) = \frac{81}{25}$

$a_n = 25\left(-\frac{3}{5}\right)^{n-1},\ r = -\frac{3}{5}$

66. $a_1 = 18,\ a_{k+1} = \frac{5}{3}a_k$

$a_1 = 18$

$a_2 = \frac{5}{3}(18) = 30$

$a_3 = \frac{5}{3}(30) = 50$

$a_4 = \frac{5}{3}(50) = \frac{250}{3}$

$a_5 = \frac{5}{3}\left(\frac{250}{3}\right) = \frac{1250}{9}$

$a_n = 18\left(\frac{5}{3}\right)^{n-1}$

$r = \frac{5}{3}$

67. $a_2 = a_1 r$

$-8 = 16r$

$-\frac{1}{2} = r$

$a_n = 16\left(-\frac{1}{2}\right)^{n-1}$

$\sum_{n=1}^{20} 16\left(-\frac{1}{2}\right)^{n-1} = 16\left[\frac{1 - (-1/2)^{20}}{1 - (-1/2)}\right] \approx 10.67$

68. $a_4 = a_3 r$

$1 = 6r \implies r = \frac{1}{6}$

$a_3 = a_1 r^2 \implies 6 = a_1\left(\frac{1}{6}\right)^2 \implies a_1 = 6^3 = 216$

$a_n = a_1 r^{n-1} = 216\left(\frac{1}{6}\right)^{n-1}$

$\sum_{n=1}^{20} 216\left(\frac{1}{6}\right)^{n-1} = 216\,\frac{1 - (1/6)^{20}}{1 - (1/6)} = 259.2$

69. $a_1 = 100,\ r = 1.05$

$a_n = 100(1.05)^{n-1}$

$\sum_{n=1}^{20} 100(1.05)^{n-1} = 100\left[\frac{1 - (1.05)^{20}}{1 - 1.05}\right] \approx 3306.60$

70. $a_1 = 5,\ r = 0.2$

$a_n = a_1\, r^{n-1} = 5\left(\frac{1}{5}\right)^{n-1}$

$\sum_{n=1}^{20} 5\left(\frac{1}{5}\right)^{n-1} = 5\left[\frac{1 - (1/5)^{20}}{1 - 1/5}\right] = 6.25$

71. $\sum_{i=1}^{7} 2^{i-1} = \frac{1 - 2^7}{1 - 2} = 127$

72. $\sum_{i=1}^{5} 3^{i-1} = \frac{1 - 3^5}{1 - 3} = 121$

73. $\sum_{n=1}^{7} (-4)^{n-1} = \frac{1 - (-4)^7}{1 - (-4)}$

$= 3277$

74. $\sum_{n=1}^{4} 12\left(-\frac{1}{2}\right)^{n-1} = 7.5$

75. $\sum_{n=0}^{4} 250(1.02)^n = 250\left(\frac{1 - 1.02^5}{1 - 1.02}\right) = 1301.01004$

76. $\sum_{n=0}^{5} 400(1.08)^n \approx 2934.3716$

77. $\sum_{i=1}^{10} 10\left(\frac{3}{5}\right)^{i-1} \approx 24.849$

78. $\sum_{i=1}^{15} 20(0.2)^{i-1} \approx 25$

79. $\sum_{i=1}^{\infty} 4\left(\frac{7}{8}\right)^{i-1} = \sum_{i=0}^{\infty} 4\left(\frac{7}{8}\right)^{i} = \frac{4}{1 - 7/8} = 32$

80. $\sum_{i=1}^{\infty} 6\left(\frac{1}{3}\right)^{i-1} = \sum_{i=0}^{\infty} 6\left(\frac{1}{3}\right)^{i} = \frac{6}{1 - 1/3} = 9$

81. $\sum_{k=1}^{\infty} 4\left(\frac{2}{3}\right)^{k-1} = \frac{4}{1 - 2/3} = 12$

82. $\sum_{k=1}^{\infty} 1.3\left(\frac{1}{10}\right)^{k-1} = \frac{1.3}{1 - (1/10)} = \frac{13}{9}$

83. (a) $a_t = 120{,}000(0.7)^t$

(b) $a_5 = 120{,}000(0.7)^5 = \$20{,}168.40$

84. $A = \sum_{i=1}^{48} 75\left(1 + \frac{0.04}{12}\right)^{i} = \$3909.96$

85. 1. When $n = 1$, $2 = \frac{1}{2}(5(1) - 1)$.

2. Assume that $S_k = 2 + 7 + \cdots + (5k - 3) = \frac{k}{2}(5k - 1)$. Then,

$$
\begin{aligned}
S_{k+1} &= 2 + 7 + \cdots + (5k - 3) + [5(k + 1) - 3] \\
&= S_k + 5k + 2 \\
&= \frac{k}{2}(5k - 1) + 5k + 2 \\
&= \frac{1}{2}[5k^2 + 9k + 4] \\
&= \frac{1}{2}[(5k + 4)(k + 1)] \\
&= \frac{k + 1}{2}(5(k + 1) - 1).
\end{aligned}
$$

Therefore, by mathematical induction, the formula is true for all positive integers n.

86. 1. When $n = 1$, $1 = \frac{1}{4}(1 + 3) = 1$.

2. Assume that $1 + \frac{3}{2} + 2 + \frac{5}{2} + \cdots + \frac{1}{2}(k + 1) = \frac{k}{4}(k + 3)$. Then,

$$
\begin{aligned}
1 + \frac{3}{2} + 2 + \frac{5}{2} + \cdots + \frac{1}{2}(k + 1) + \frac{1}{2}(k + 2) &= \frac{k}{4}(k + 3) + \frac{1}{2}(k + 2) \\
&= \frac{k(k + 3) + 2(k + 2)}{4} \\
&= \frac{k^2 + 5k + 4}{4} \\
&= \frac{(k + 1)(k + 4)}{4} \\
&= \frac{k + 1}{4}[(k + 1) + 3].
\end{aligned}
$$

Thus, the formula holds for all positive integers n.

87. 1. When $n = 1$, $a = a\left(\frac{1-r}{1-r}\right)$.

2. Assume that

$$S_k = \sum_{i=0}^{k-1} ar^i = \frac{a(1-r^k)}{1-r}.$$

Then,

$$S_{k+1} = \sum_{i=0}^{k} ar^i = \sum_{i=0}^{k-1} ar^i + ar^k = \frac{a(1-r^k)}{1-r} + ar^k$$

$$= \frac{a(1-r^k+r^k-r^{k+1})}{1-r} = \frac{a(1-r^{k+1})}{1-r}.$$

Therefore, by mathematical induction, the formula is valid for all positive integer values of n.

88. 1. When $n = 1$, $a + 0 \cdot d = a = \frac{1}{2}[2a + (1-1)d] = a$.

2. Assume that $\sum_{k=0}^{i-1}(a + kd) = \frac{i}{2}[2a + (i-1)d]$, using i as the induction variable. Then,

$$\sum_{k=0}^{i+1-1}(a+kd) = \frac{i}{2}[2a + (i-1)d] + [a + id]$$

$$= \frac{2ia + i(i-1)d + 2a + 2id}{2} = \frac{2a(i+1) + id(i+1)}{2} = \left(\frac{i+1}{2}\right)[2a + id].$$

Thus, the formula holds for all positive integers n.

89. $\sum_{n=1}^{30} n = \frac{30(31)}{2} = 465$

90. $\sum_{n=1}^{10} n^2 = \frac{10(10+1)(20+1)}{6} = 385$

91. $\sum_{n=1}^{7}(n^4 - n) = \sum_{n=1}^{7} n^4 - \sum_{n=1}^{7} n$

$$= \frac{7(8)(15)[3(7)^2 + 3(7) - 1]}{30} - \frac{7(8)}{2}$$

$$= \frac{840(167)}{30} - 28 = 4676 - 28 = 4648$$

92. $\sum_{n=1}^{6}(n^5 - n^2) = \frac{6^2(7^2)(2 \cdot 6^2 + 12 - 1)}{12} - \frac{6(7)(2(6)+1)}{6} = 12{,}201 - 91 = 12{,}110$

93. $a_1 = f(1) = 5$

$a_2 = a_1 + 5 = 5 + 5 = 10$

$a_3 = a_2 + 5 = 15$

$a_4 = a_3 + 5 = 20$

$a_5 = a_4 + 5 = 25$

	n:	1	2	3	4	5
	a_n:	5	10	15	20	25
First differences:			5	5	5	5
Second difference:				0	0	0

Linear model: $a_n = 5n$

94. $a_1 = f(1) = -3$

$a_2 = a_1 - 2(2) = -3 - 4 = -7$

$a_3 = -7 - 2(3) = -7 - 6 = -13$

$a_4 = -13 - 2(4) = -13 - 8 = -21$

$a_5 = -21 - 2(5) = -21 - 10 = -31$

n:	1	2	3	4	5
a_n:	-3	-7	-13	-21	-31
First differences:		-4	-6	-8	-10
Second differences:			-2	-2	-2

Quadratic model

95. $a_1 = f(1) = 16$

$a_2 = a_1 - 1 = 16 - 1 = 15$

$a_3 = a_2 - 1 = 15 - 1 = 14$

$a_4 = 14 - 1 = 13$

$a_5 = 13 - 1 = 12$

	n:	1	2	3	4	5
	a_n:	16	15	14	13	12
First differences:			-1	-1	-1	-1
Second difference:				0	0	0

Linear model: $a_n = 17 - n$

96. $a_1 = f(1) = 1$

$a_2 = 2 - a_1 = 2 - 1 = 1$

$a_3 = 3 - a_2 = 2$

$a_4 = 4 - 2 = 2$

$a_5 = 5 - 2 = 3$

n:	1	2	3	4	5
a_n:	1	1	2	2	3
First differences:		0	1	0	1
Second differences:			1	-1	1

Neither linear nor quadratic

97. ${}_{10}C_8 = 45$

98. ${}_{12}C_5 = 792$

99. $\binom{9}{4} = {}_9C_4 = 126$

100. $\binom{14}{12} = {}_{14}C_{12} = 91$

101. 4th number in 6th row is ${}_6C_3 = 20$.

102. The 8th entry in the 9th row is 36.

103. 5th number in 8th row is $\binom{8}{4} = {}_8C_4 = 70.$

104. The 6th entry in the 10th row is 252.

105. $(x + 5)^4 = x^4 + 4x^3(5) + 6x^2(5^2) + 4x(5^3) + 5^4$
$= x^4 + 20x^3 + 150x^2 + 500x + 625$

106. $(y - 3)^3 = y^3 - 9y^2 + 27y - 27$

107. $(a - 4b)^5 = a^5 - 5a^4(4b) + 10a^3(4b)^2 - 10a^2(4b)^3 + 5a(4b)^4 - (4b)^5$
$= a^5 - 20a^4b + 160a^3b^2 - 640a^2b^3 + 1280ab^4 - 1024b^5$

108. $(3x + y)^7 = 2187x^7 + 5103x^6y + 5103x^5y^2 + 2835x^4y^3 + 945x^3y^4 + 189x^2y^5 + 21xy^6 + y^7$

109. $(7 + 2i)^4 = 7^4 + 4(7)^3(2i) + 6(7)^2(2i)^2 + 4(7)(2i)^3 + (2i)^4$
$= 2401 + 2744i - 1176 - 224i + 16$
$= 1241 + 2520i$

110. $(4 - 5i)^3 = 4^3 - 3(4)^2(5i) + 3(4)(5i)^2 - (5i)^3$
$= 64 - 240i - 300 + 125i$
$= -236 - 115i$

111. $E = \{(1, 11), (2, 10), (3, 9), (4, 8), (5, 7), (7, 5), (8, 4), (9, 3), (10, 2), (11, 1)\}$
$n(E) = 10$

112. $(2!)(6!) = 1440$ ways

113. (a) $(4)(3)(6)(3) = 216$ schedules
(b) $(2)(3)(6)(3) = 108$ schedules
(c) $(2)(3)(2)(3) = 36$ schedules

114. (a) $10^7 = 10{,}000{,}000$ possible calls
(b) $2 \cdot 10^6 = 2{,}000{,}000$ calls
(c) $10{,}000{,}000 - 2{,}000{,}000 = 8{,}000{,}000$ calls

115. ${}_{10}C_8 = \dfrac{10!}{2!8!} = \dfrac{10 \cdot 9}{2} = 45$

116. ${}_8C_6 = \dfrac{8!}{2!6!} = \dfrac{8 \cdot 7}{2} = 28$

117. ${}_{12}P_{10} = \dfrac{12!}{2!} = 239{,}500{,}800$

118. ${}_6P_4 = \dfrac{6!}{2!} = 360$

119. ${}_{100}C_{98} = \dfrac{100!}{2!98!} = \dfrac{100 \cdot 99}{2} = 4950$

120. ${}_{50}C_{48} = \dfrac{50!}{2!48!} = \dfrac{50 \cdot 49}{2} = 1225$

121. ${}_{1000}P_2 = \dfrac{1000!}{998!} = 1000(999) = 999{,}000$

122. ${}_{500}P_2 = \dfrac{500!}{498!} = 500(499) = 249{,}500$

123. $\dfrac{8!}{2!2!2!1!1!} = \dfrac{8!}{8} = 7! = 5040$ permutations

124. $\dfrac{9!}{2!2!} = \dfrac{9!}{4} = 90{,}720$ permutations

125. $10! = 3{,}628{,}800$ ways

126. $({}_7C_2)({}_{11}C_2) = 21 \cdot 55 = 1155$ ways

127. ${}_{20}C_{15} = 15{,}504$ ways

128. ${}_{54}C_6 = 25{,}827{,}165$ ways

129.
$$\begin{aligned} {}_{n+1}P_2 &= 4 \cdot {}_nP_1 \\ \frac{(n+1)!}{(n-1)!} &= 4 \cdot \frac{n!}{(n-1)!} \\ (n+1)! &= 4 \cdot n! \\ n &= 3 \end{aligned}$$

130.
$$\begin{aligned} 8 \cdot {}_nP_2 &= {}_{n+1}P_3 \\ 8\frac{n!}{(n-2)!} &= \frac{(n+1)!}{(n-2)!} \\ 8n! &= (n+1)! \\ n &= 7 \end{aligned}$$

131. $\frac{10}{10} \cdot \frac{1}{9} = \frac{1}{9}$

132. $P(E) = \frac{n(E)}{n(S)} = \frac{1}{5!} = \frac{1}{120}$

133. (a) $\frac{208}{500} = 0.416$

(b) $\frac{400}{500} = 0.8$

(c) $\frac{37}{500} = 0.074$

134. $\left(\frac{6}{6}\right)\left(\frac{5}{6}\right)\left(\frac{4}{6}\right)\left(\frac{3}{6}\right)\left(\frac{2}{6}\right)\left(\frac{1}{6}\right) = \frac{6!}{6^6} = \frac{720}{46{,}656} = \frac{5}{324}$

135. $P(2 \text{ pairs}) = \frac{({}_{13}C_2)({}_4C_2)({}_4C_2)({}_{44}C_1)}{({}_{52}C_5)} = 0.0475$

136. $P(\text{club}) = \frac{13}{52} = \frac{1}{4}$

$P(\text{not club}) = 1 - \frac{1}{4} = \frac{3}{4}$

137. True

$$\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$$

138. True

139. Answers will vary. See pages 526 and 535.

140. They differ by a minus sign.

(a) $-1, \frac{1}{2}, -\frac{1}{3}, \ldots$
(Odd-numbered terms are negative.)

(b) $1, -\frac{1}{2}, \frac{1}{3}, \ldots$
(Even-numbered terms are negative.)

141. (a) Arithmetic-linear model

(b) Geometric model

142.
$$\begin{aligned} S_6 &= 130 + 70 + 40 = 240 \\ S_7 &= 240 + 130 + 70 = 440 \\ S_8 &= 440 + 240 + 130 = 810 \\ S_9 &= 810 + 440 + 240 = 1490 \\ S_{10} &= 1490 + 810 + 440 = 2740 \end{aligned}$$

143. Answers will vary. See page 528. To define a sequence recursively, you need to be given one or more of the first few terms. All other terms are defined using previous terms.

144. When $0 < r < 1$, $a_n = a_{n-1}(r) < a_{n-1}$.

145. If n is even, the expansion are the same. If n is odd, the expansion of $(-x + y)^n$ is the negative of that of $(x - y)^n$.

146. In the closed interval $[0, 1]$.

Chapter 8 Practice Test

1. Write out the first five terms of the sequence $a_n = \dfrac{2n}{(n+2)!}$.

2. Write an expression for the nth term of the sequence $\left\{\frac{4}{3}, \frac{5}{9}, \frac{6}{27}, \frac{7}{81}, \frac{8}{243}, \ldots\right\}$.

3. Find the sum $\displaystyle\sum_{i=1}^{6}(2i-1)$.

4. Write out the first five terms of the arithmetic sequence where $a_1 = 23$ and $d = -2$.

5. Find a_{50} for the arithmetic sequence with $a_1 = 12$, $d = 3$, and $n = 50$.

6. Find the sum of the first 200 positive integers.

7. Write out the first five terms of the geometric sequence with $a_1 = 7$ and $r = 2$.

8. Evaluate $\displaystyle\sum_{n=0}^{9} 6\left(\frac{2}{3}\right)^n$.

9. Evaluate $\displaystyle\sum_{n=0}^{\infty} (0.03)^n$.

10. Use mathematical induction to prove that $1 + 2 + 3 + 4 + \cdots + n = \dfrac{n(n+1)}{2}$.

11. Use mathematical induction to prove that $n! > 2^n$, $n \geq 4$.

12. Evaluate ${}_{13}C_4$. Verify with a graphing utility.

13. Expand $(x+3)^5$.

14. Find the term involving x^7 in $(x-2)^{12}$.

15. Evaluate ${}_{30}P_4$.

16. How many ways can six people sit at a table with six chairs?

17. Twelve cars run in a race. How many different ways can they come in first, second, and third place? (Assume that there are no ties.)

18. Two six-sided dice are tossed. Find the probability that the total of the two dice is less than 5.

19. Two cards are selected at random from a deck of 52 playing cards without replacement. Find the probability that the first card is a King and the second card is a black ten.

20. A manufacturer has determined that for every 1000 units it produces, 3 will be faulty. What is the probability that an order of 50 units will have one or more faulty units?